PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Figure of the Earth from Gravity Observations and the Precision Obtainable

J. de Graaff Hunter

Phil. Trans. R. Soc. Lond. A 1935 **234**, 377-431
doi: 10.1098/rsta.1935.0012

XIII—The Figure of the Earth from Gravity Observations and the Precision Obtainable

By J. de Graaff Hunter, *C.I.E.*, *Sc.D.*

(Communicated by Sir Gerald Lenox-Conyngham, F.R.S.—Received September 24, 1934. Read January 24, 1935)

Contents

Introduction

In 1849 Stokes published a remarkable relation between the form of the geoid and the values of gravity. He neglected terms involving the square of the ellipticity. The validity of his expression for the external potential has been doubted by some later writers, particularly for purposes of a higher approximation.

Sir George Darwin, ignoring the departure of the geoid from spheroidal form, derived expressions for the internal and external potentials of the earth, keeping terms of the order of the square of the ellipticity. He justified his results for the region between two spheres concentric with the earth of radii equal to the earth's minimum and maximum radii. But again some doubted the validity of his expressions for this very region.

In the present paper the external potential is derived directly (§§ 1–13) from an extension of a theorem due to GREEN, without any assumption as to its form. The expression includes terms involving the square of the ellipticity and also the higher harmonics representing the departure of the geoid from a spheroid ; but products of these departures and the ellipticity are neglected.

The results confirm STOKES's formula, which it extends to the second order of small quantities ; and also DARWIN's work. Numerical coefficients in the derived formula for gravity agree to the last figure with those of the formula adopted at Stockholm by the International Union of Geodesy and Geophysics, § 21, details of the derivation of which have not been published so far as the writer is aware. These latter may, or may not, have remained open to the objections to which earlier results were on occasion subjected.

In §§ 14–17 the level surfaces external to the geoid are considered and the relation between mean radii and ellipticities of these surfaces is found (16.2) ; while the variation of gravity with height is expressed by (17.4).

Provided with the basic formulæ firmly established, STOKES's process of expressing the geoidal departure from a spheroid of reference gives a formula for the separation of geoid and spheroid as a solid angle integral of the gravity anomalies multiplied by a function of the angular distance, § 20. To this is added the equally important expression for the deviation of the geoidal vertical from the spheroidal.

The difficulties of topography external to the geoid are avoided by the use of gravity data reduced on a basis of compensation, deemed to relate to a corresponding compensated geoid. Return to the actual geoid is subsequently possible, when all traces of assumption of compensation disappear.

To employ the formulæ for the form of the compensated geoid, values of gravity properly distributed over the world are required. Although the number of gravity stations is now considerable, there are still great regions where no values of gravity have been measured. It is no longer impractical to suggest that this deficiency be made good, for, as will be seen, under 1700 stations would suffice for a general survey. During the writing of this paper, HIRVONEN has published a paper* giving the form of the geoid, derived from existing data, which he naturally considers to be inadequate. The paper is valuable, in the opinion of the present writer, as indicating very plainly the dearth of gravity results in many regions. Naturally no attempt is made in it to determine the deviation of the geoidal vertical, for which the data are wholly inadequate.

In the present paper the probable error of representation of one gravity station for a surrounding area has been assessed, and a semi-empirical formula for it has been derived, §§ 29, 30. It appears that the horizontal gravity gradient may be considered to have a random distribution. The probable errors of derived geoidal anomalies of height and tilt are then expressed for the case of one gravity station for each area equal to that of a five-degree square of latitude and longitude at the

* 'Veröff. finn. geod. Inst.' No. 19 (1934).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

equator ; which leads to a sub-division of the whole surface of the earth into 1654 regions. The values of gravity at 1654 points are called for. For the determination of tilt at any point, a special local survey, involving from 100 to 200 stations, is needed. Here the spacing of stations is of great importance, and the optimum distribution is derived, § 41, from the principle that each gravity station should make the same contribution to the total probable error of result. In § 46 the probable errors are discussed and found to be of a suitable order of magnitude for the purpose of deciding the geoidal elements at the origin of a survey and for the purpose of computing a terrestrial arc for reduction of observations of lunar parallax. With the 1654 evenly distributed stations and (134 — 50 = 84) additional local stations, distributed to best advantage, the probable error of direction of the vertical is 0″·35 (*vide* § 44) and the probable error in elevation of the geoid is 23 feet (*vide* § 45).

The paper terminates with some remarks on the practical application of observational results to the determination of the geoidal elements, §§ 47–49.

It is hoped that by necessary international co-operation the deficiencies in our knowledge of gravity anomalies in many regions may be made good in the course of the next decade ; it will then be possible to express the results of the main geodetic surveys in terms of a unique reference figure, and to be content no longer with rather vague estimates as to the possible departure of the earth's sea-level surface (geoid) from what it is ordinarily assumed to be.

In conclusion, I gratefully acknowledge the continued interest and helpful counsel of Sir Gerald Lenox-Conyngham during the writing of this paper.

Notation used in 1–17

Astronomical unit of mass is employed ; so $g = - dV/dn$, where n is the outward drawn normal.

Reference spheroid I is $r = a\,(1 - \frac{2}{3}\varepsilon P_2) = a'\,(1 - \varepsilon' \cos^2\theta)$.
Reference spheroid II is $r = a''\,(1 - \varepsilon\cos^2\theta - \frac{3}{2}\varepsilon^2\cos^2\theta\sin^2\theta)$.
The compensated geoid is $r = a\,(1 + u) = a\,(1 - \frac{2}{3}\varepsilon P_2 + \Sigma\, u_n)$

$$= a'\,\{1 - \varepsilon'\cos^2\theta + \Sigma\,(u_n - u''_n)\}$$

In the above P_2 is the Legendre function $\frac{1}{2}\,(3\cos^2\theta - 1)$, θ being measured from the earth's pole of rotation.

u_n is a Laplace function of order n.
u''_n is the same function when $\theta = 90°$, *i.e.*, at the equator.
ε' is an adopted value of the ellipticity, derived from existing data and so liable to modification, by amount of order ε^2.
a = mean radius of the compensated geoid
= mean radius of the geoid.
a' = equatorial radius of spheroid I
= mean equatorial radius of the compensated geoid, subject to observational error.

The value of a is derived from existing triangulation. It cannot be found so accurately from values of gravity alone, nor is a highly precise value required for discussion of gravity values. An error in a (or a') causes a corresponding error in m or m', which are of same order as ε.

G = mean gravity over the compensated geoid.
G′ = equatorial gravity of spheroid I, supposed to be a level surface of internal mass E and rotation ω.
G″ = equatorial gravity of compensated geoid, depending on longitude.
$\gamma = G(1 + \alpha P_2 + \beta P_4)$ = gravity at any point of spheroid I.
γ_s = gravity at any point of spheroid II.
α, β are constants, *vide* (11.7), (11.8), involving m', ε.
g = actual compensated gravity at any point external to compensated geoid.
g_0 = value of g on the compensated geoid $= \gamma + Gv$.
$v = \Sigma v_n$, where v_n is a Laplace function of order n.
$\Delta g_0 = g_0 - \gamma = Gv$.
$\Delta g_s = g_0 - \gamma_s$.
E = mass of the earth in astronomical units.
ω = angular velocity of the earth.
$\omega^2 a'/G' = m = 1/288{\cdot}361$.
$\omega^2 a/G = m'$.
$\omega^2 a^3/G = k$.
V = combined potential of the compensated geoid and its rotation.
V_c = value of V on the compensated geoid.
V″ = attractional potential of the compensated geoid.
V′ = attractional potential of the geoid.
U = potential of rotation.
S is a closed surface, but after § 5 it represents the compensated geoid.
ψ, χ are angular co-ordinates, reckoned from a point P′ (*vide* fig. § 8).
P′ is the point at which geoidal anomalies are sought.
$p = \cos\theta$.
λ, μ, ν, *see* (9.10).
η = inclination of spheroidal normal to radius vector, *see* (9.5).
ϕ = latitude.

The Combined Gravitational and Rotational Potential and the Force of Gravity at the Earth's (Compensated) Sea-Level Surface or Geoid

1—In an "Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" published at Nottingham in 1828,* Green showed that if V, V′ are two continuous functions of the rectangular co-ordinates x, y, z, whose

* "Mathematical Papers of the late George Green," edited by N. M. Ferrers; Macmillan & Co. (1871).

differential coefficients do not become infinite at any point within a solid body of any form whatever, then

$$\int V \frac{dV'}{dn} d\sigma + 4\pi \int V\rho' \, dw = \int V' \frac{dV}{dn} d\sigma + 4\pi \int V'\rho \, dw, \quad . \; . \; . \quad (1.1)$$

where

$4\pi\rho = -\nabla^2 V,$

$4\pi\rho' = -\nabla^2 V',$

$d\sigma$ is an element of surface of the closed surface S,

dw is an element of volume of the closed surface S,

dn is an element of the outward drawn normal to S

and the integrations are taken throughout the volume of S as regards dw and over its entire surface as regards $d\sigma$. Green showed further that if V is the attractional potential of a system of masses, partly within and partly without S, then putting $V' = 1/R'$, where R' is the distance of any point within S from a point P', so that $\rho' = 0$, and making S an equipotential of the mass system on which $V = V_c$, (1.1) becomes

$$V_c \int \frac{d}{dn}\left(\frac{1}{R'}\right) d\sigma = \int \frac{dV}{dn}\frac{d\sigma}{R'} + 4\pi \int \frac{\rho}{R'} \, dw. \quad . \; . \; . \; . \; . \; . \quad (1.2)$$

Now $1/R'$ is the potential at $d\sigma$ due to a unit mass at P', so that $\int \frac{d}{dn}\left(\frac{1}{R'}\right) d\sigma = 0$ when P' is outside S, for which case $V' = 1/R'$ never becomes infinite. So (1.2) may be written

$$\begin{pmatrix}\text{Potential at exterior} \\ \text{point } P' \text{ of mass within S}\end{pmatrix} = \int \frac{\rho}{R'} \, dw = -\frac{1}{4\pi}\int \frac{dV}{dn}\frac{d\sigma}{R'}. \quad . \; . \; . \; . \quad (1.3)$$

This implies that if a thin layer of matter of density $\left(-\frac{1}{4\pi}\frac{dV}{dn}\right)$ is placed on the surface S, the potential of this layer at any *external* point P' is the same as the potential at P' of the matter within S.

It was also shown by Green that if V' becomes infinite at a point P' in the region of the integration that a term $4\pi V_1$ must be added to the left hand of (1.1), V_1 being the value of V at P_1. So for an *internal* point P_1, for which by Gauss's theorem $\int \frac{d}{dn}\left(\frac{1}{R'}\right) d\sigma = -4\pi$, equation (1.2) becomes

$$-4\pi V_c + 4\pi V_1 = \int \frac{dV}{dn}\frac{d\sigma}{R'} + 4\pi \int \frac{\rho}{R'} \, dw,$$

or

$$V_1 - \int \frac{\rho}{R'} \, dw = V_c + \frac{1}{4\pi}\int \frac{dV}{dn}\frac{d\sigma}{R'}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.4)$$

So

$$\left(\text{Potential at } P_1 \text{ of surface density} - \frac{1}{4\pi}\frac{dV}{dn}\right) + \left(\begin{matrix}\text{Potential at } P_1 \text{ of} \\ \text{matter exterior to S}\end{matrix}\right)$$
$$= (\text{potential of whole system over S}). \quad (1.5)$$

2—A more comprehensive result, useful for treating the figure of the earth, is obtained if we suppose that

$$V = V'' + U, \quad \ldots \ldots \ldots \ldots \quad (2.1)$$

where V'' is the potential of attractional matter both within and without S and U is the potential of any conservative system of forces ; S being still an equipotential of V, which for distinction I call a " level surface." In this case

$$-4\pi\rho = \nabla^2 V'' + \nabla^2 U = -4\pi\tau + \nabla^2 U, \quad \ldots \ldots \quad (2.2)$$

in which τ is the density of attractional matter. Equation (1.2) may now be written for *exterior* point P′

$$\int \frac{\tau}{R'}\,dw + \int \frac{1}{4\pi}\frac{dV}{dn}\frac{d\sigma}{R'} - \int \frac{\nabla^2 U}{4\pi R'}\,dw = 0, \quad \ldots \ldots \quad (2.3)$$

or

(potential at exterior point P′ of matter within S)

$$= \left(\text{potential at P}' \text{ of surface density} - \frac{1}{4\pi}\frac{dV}{dn}\right) + \int \frac{\nabla^2 U}{4\pi R'}\,dw. \quad \ldots \quad (2.4)$$

For an *interior* point P_1, from (1.4) it follows that

$$V''_1 - \int \frac{\tau}{R'}\,dw + U_1 + \frac{1}{4\pi}\int \frac{\nabla^2 U}{R'}\,dw = V_c + \int \frac{1}{4\pi}\frac{dV}{dn}\frac{d\sigma}{R'}, \quad \ldots \quad (2.5)$$

or

$$\left(\text{potential at } P_1 \text{ of surface density} - \frac{1}{4\pi}\frac{dV}{dn}\right) + \left(\begin{matrix}\text{potential at } P_1 \text{ of} \\ \text{matter exterior to S}\end{matrix}\right)$$
$$= V_c - U_1 - \frac{1}{4\pi}\int \frac{\nabla^2 U}{R'}\,dw. \quad \ldots \ldots \ldots \quad (2.6)$$

3—For the case of a uniformly rotating body, such as the earth, $U = \frac{1}{2}\omega^2(x^2 + y^2)$, x, y, z, being rectangular co-ordinates referred to axes of which the z-axis coincides with the axis of rotation, and ω being the angular velocity. Then

$$\nabla^2 U = 2\omega^2, \quad \ldots \ldots \ldots \ldots \quad (3.1)$$

and

$$\int \frac{\nabla^2 U}{4\pi R'}\,dw = \frac{\omega^2}{2\pi}\int \frac{dw}{R'} = \frac{\omega^2}{2\pi}\,(\text{potential of unit density throughout S}). \quad (3.2)$$

Hence at a point external to or on S

$$\text{Potential at P}' \text{ of matter within S} = \text{potential of surface density } g/4\pi$$
$$+ \frac{\omega^2}{2\pi} \text{(potential of unit density throughout S)}, \quad (3.3)$$

where

$$g = - dV/dn = \text{normal force at surface S due to attraction and rotation.}$$

4—Equation (3.3) allows the potential of all matter within S, which in the case of the figure of the earth is the geoid, to be found without any assumption whatever as to the distribution of internal density, provided that the value of gravity at geoidal level is known at sufficient points. There is no occasion to *assume* the form of the external potential, as done by STOKES,* DARWIN† and others. It will be seen, however, that the form they assumed is justifiable (as indeed many believed).

5—The potential of that portion of the earth which extends outside the geoid can also be calculated if its density from ground level down to the geoid can be reasonably assumed. Combining the result with the potential of the interior matter as given by (3.3) we obtain the complete potential. Observed values of gravity need similar reduction for external topography. It is, however, simpler to get rid of the topography, which can be done by use of the " compensated geoid." This surface differs from the actual geoid by the calculable effect of the compensated topography ; and the computation is easy.‡ The observed values of gravity can similarly be corrected for compensated topography, as indeed is often done on the basis of the HAYFORD system, and is necessarily on the same system as is employed in deriving the compensated geoid. Without discussing the details further at this point it will suffice to say that the effect of the matter outside the geoid will be removed by some practical method of computation, the gravity data being correspondingly reduced ; and we proceed to consider the case of a rotating earth *bounded* by the compensated geoid which is its sea-level surface. For this (3.3) needs no additional term.

6—In a system of polar co-ordinates any point on a surface having no re-entrants and enclosing the origin is uniquely defined by the equation of the surface and the two angular co-ordinates. We may express the form, which does not differ widely from that of a sphere, of the compensated geoid by the equation

$$r = a(1 + u), \qquad (6.1)$$

where u is a harmonic function of the angular co-ordinates. In the same way the value of the force of attraction and rotation acting normally to (6.1) at any point on that surface may be expressed by

$$g_0 = G(1 + v). \qquad (6.2)$$

* 'Math. and Phys. Papers,' vol. 2, p. 131, Camb. Univ. Press (1883).
† 'Mon. Not. R. astr. Soc.,' vol. 60, p. 81 (1900).
‡ DE GRAAFF HUNTER and BOMFORD, 'Bull. géod. int.,' No. 29 (1931).

Both u and v may be expressed uniquely as series of LAPLACE functions—surface spherical harmonics; and a and G are mean values of r and g_0 over the whole compensated geoid. Geodetic measurements at numerous points of the earth's surface have indicated that the geoid approximates to a spheroid of revolution whose equation may be represented sufficiently closely by

$$\left.\begin{array}{l} r = a'(1-\varepsilon'\cos^2\theta) = a'(1-\tfrac{1}{3}\varepsilon'-\tfrac{2}{3}\varepsilon' P_2) = a(1-\tfrac{2}{3}\varepsilon P_2) \\ \text{where} \quad a = a'(1-\tfrac{1}{3}\varepsilon') \qquad a' = a(1+\tfrac{1}{3}\varepsilon) \\ \qquad \varepsilon = \varepsilon'/(1-\tfrac{1}{3}\varepsilon') \qquad \varepsilon' \doteqdot \varepsilon(1-\tfrac{1}{3}\varepsilon) \end{array}\right\}, \quad (6.3)$$

where P_2 is the LEGENDRE function $(3p^2-1)/2$ and $p = \cos\theta$ and ε' is (conventionally) the ellipticity, of value about 1/300. The deviations of the geoid from spheroidal form have been determined only in detached and relatively small regions. Thus in India the deviations from the best fitting spheroid over about a million square miles (0·5% of the earth's surface) are comprised within about 60 feet or $3\times10^{-6}a$. It is supposed—at least until the reverse is found—that the greatest deviation from spheroidal form is of the order of some few hundreds of feet, and for the present enquiry we consider that the deviation is of the order ε^2. Accordingly we put

$$u = -\tfrac{2}{3}\varepsilon P_2 + \sum_2^\infty u_n, \quad \ldots \quad (6.4)$$

in which u_n is of the order of magnitude ε^2. The first harmonic does not occur when the origin is chosen to coincide with the centre of gravity of the earth. It is to be remarked that practically a value of ε dependent on existing observations must be employed. The term u_2 will contain a small term $+\tfrac{2}{3}\delta\varepsilon P_2$, where $\delta\varepsilon$ is the error in the adopted value of ε.

7—On the surface S, that is the compensated geoid, the combined potential of rotation and attraction is constant. It is, by (3.3) and (2.1)

$$V_c = \tfrac{1}{2}\omega^2 r^2\sin^2\theta + \frac{\omega^2}{2\pi}\int\frac{dw}{R'} + \frac{1}{4\pi}\int\frac{g_0 d\sigma}{R'}. \quad \ldots \quad (7.1)$$

The ratio of centrifugal force at the equator to equatorial gravity is usually denoted by m whose approximate value is 1/288 which is of the same order as ε. So

$$m = \omega^2 a'/G', \quad \ldots \quad (7.2)$$

We define G′ as the equatorial value of gravity on the spheroid. It will be more convenient at present to use

$$m' = \omega^2 a/G, \quad \ldots \quad (7.3)$$

in which m' is slightly different from m, but of nearly the same value. Then (7.1) becomes

$$\begin{aligned} V_c &= \frac{m'}{2}\frac{r^2}{a}G\sin^2\theta' + \frac{m'G}{2\pi a}\int\frac{dw}{R'} + \frac{G}{4\pi}\int\frac{(1+v)d\sigma}{R'} \\ &= U + W + X, \quad \ldots \quad (7.4) \end{aligned}$$

and this is constant when

$$r = a\,(1 - \tfrac{2}{3}\varepsilon\, P_2 + \sum_2^\infty u_n). \qquad (7.5)$$

8—It is convenient to note that if $p = \cos\,\theta$

$$P_2 = (3\,p^2 - 1)/2 \qquad P_4 = (35\,p^4 - 30\,p^2 + 3)/8$$
$$= 35\,(P_2{}^2 - \tfrac{2}{7}P_2 - 1/5)/18 \qquad (8.1)$$
$$P_2{}^2 = 18P_4/35 + 2P_2/7 + 1/5 \qquad (8.2)$$
$$3\cos^2\theta = 1 + 2P_2 \qquad 3\sin^2\theta = 2\,(1 - P_2)$$
$$\sin^2 2\theta = 8\,(7 + 5P_2 - 12P_4)/105. \qquad (8.3)$$

In fig. 1

CP $= \theta$ CP′ $= \theta'$ PP′ $= \psi$.

The first term of (7.4) is

$$U = \frac{m'}{2}\,Ga\,(1 - \tfrac{4}{3}\varepsilon P_2)\,\tfrac{2}{3}\,(1 - P_2)$$

$$U/Ga = \tfrac{1}{3}m'\,\{1 - (1 + \tfrac{4}{3}\varepsilon)\,P_2 + \tfrac{4}{3}\varepsilon P_2{}^2\}. \qquad (8.4)$$

The second term, W, of (7.4) involves $\int \frac{dw}{R'}$ which is the potential at P′ of a solid of uniform density bounded by S. As there is the small multiplier m' this need only be taken to a first approximation for which, from (6.3)

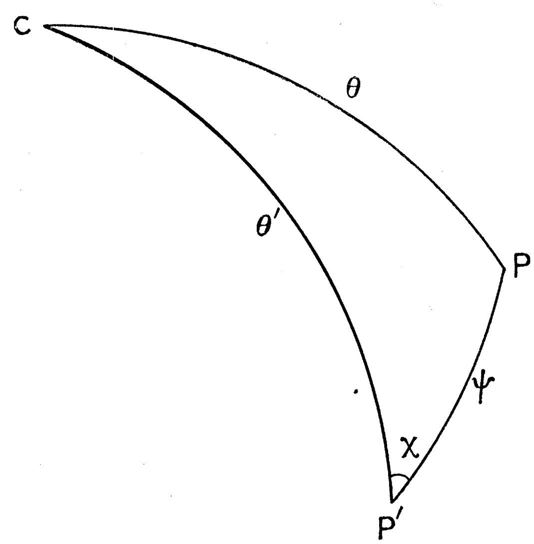

Fig. 1

$$r = a\,(1 - \varepsilon\cos^2\theta). \qquad (8.5)$$

In this case*

$$\int \frac{dw}{R'} = \tfrac{1}{2}\,(D - A\,.\,r^2\sin^2\theta - C\,.\,r^2\cos^2\theta),$$

where

$$A = \frac{4\pi}{3}\,(1 - \tfrac{2}{5}\varepsilon), \qquad C = \frac{4\pi}{3}\,(1 + \tfrac{4}{5}\varepsilon), \qquad D = 4\pi a^2. \qquad (8.6)$$

So

$$\int \frac{dw}{R'} = 2\pi a^2 - \frac{2\pi r^2}{3}\,(1 + \tfrac{4}{5}\varepsilon P_2) = \frac{4\pi a^2}{3}\,(1 + \tfrac{4}{15}\varepsilon P_2), \qquad (8.7)$$

and

$$W/Ga = \frac{2m'}{3}\,(1 + \tfrac{4}{15}\varepsilon P_2). \qquad (8.8)$$

* These results are taken from "Analytical Statics," vol. 2, p. 110, by E. J. Routh, Camb. Univ. Press.

9—The third term of (7.4)

$$= X = \frac{G}{4\pi} \int \frac{(1+v)\, d\sigma}{R'} . \qquad (9.1)$$

Now,

$$R'^2 = OP^2 + OP'^2 - 2OP \,.\, OP' \cos\psi = (OP - OP')^2 + 4\, OP \,.\, OP' \sin^2 \tfrac{1}{2}\psi$$

$$= a^2 \{(\Delta u)^2 + 4\,(1+u)\,(1+u') \sin^2 \tfrac{1}{2}\psi\}, \qquad (9.2)$$

where

$$\Delta u = u - u', \qquad (9.3)$$

and u' is the value of u at P'. Δu is a first order term, but it vanishes absolutely when $\psi = 0$, since P' is on S, and is always less than $R' \sin \frac{1}{2}\psi$. So,

$$\frac{a}{R'} = \frac{1}{2\sqrt{(1+u)\,(1+u')} \sin \frac{1}{2}\psi} \left\{1 - \frac{(\Delta u)^2}{8 \sin^2 \frac{1}{2}\psi}\right\} . \qquad (9.4)$$

Also $d\sigma = r^2 \sec\eta \, d\omega$, where $d\omega = \sin\psi d\psi d\chi$ and η is the inclination of the normal to the radius vector. An approximate value of η only is needed and so, using (8.3) we write

$$\eta \doteqdot \tan\eta = \frac{dr}{r\,d\theta} = \varepsilon \sin 2\theta, \qquad (9.5)$$

and

$$\sec\eta = 1 + \tfrac{1}{2}\varepsilon^2 \sin^2 2\theta = 1 + \frac{4\varepsilon^2}{105}(7 + 5P_2 - 12P_4). \qquad (9.6)$$

Hence

$$X = \frac{Ga}{4\pi} \int \frac{(1+v)\,(1+u)^{\frac{3}{2}}}{2 \sin \frac{1}{2}\psi\, (1+u')^{\frac{1}{2}}} \sec\eta \left\{1 - \frac{(\Delta u)^2}{8 \sin^2 \frac{1}{2}\psi}\right\} d\omega$$

$$\doteqdot \frac{Ga}{4\pi\,(1+u')^{\frac{1}{2}}} \int \frac{(1+v)\,(1+u)^{\frac{3}{2}}}{2 \sin \frac{1}{2}\psi} \sec\eta \; du - \frac{Ga}{4\pi} \int \frac{(\Delta u)^2}{16 \sin^3 \frac{1}{2}\psi} d\omega. \qquad (9.7)$$

Now, in keeping with (6.2) and (6.4) put

$$v = \alpha P_2 + \beta P_4 + \sum_2^\infty v_n, \qquad (9.8)$$

in which the first two terms arise from the term $-\frac{2}{3}\varepsilon P_2$ in (6.4) and α, β are of the first and second order respectively ; then

$$(1+v)\,(1+u)^{\frac{3}{2}} \doteqdot (1+v)\,(1 + \tfrac{3}{2}u + \tfrac{3}{8}u^2) \doteqdot 1 + v + \tfrac{3}{2}u + \tfrac{3}{2}uv + \tfrac{3}{8}u^2$$

$$= 1 - \varepsilon P_2 + \alpha P_2 + \beta P_4 + P_2{}^2\,(-\varepsilon\alpha + \tfrac{1}{6}\varepsilon^2) + \sum_2^\infty (v_n + \tfrac{3}{2}u_n). \qquad (9.8)$$

Combining (9.6) and (9.8) and substituting for P_2^2 from (8.2)

$$(1+v)(1+u)^{\frac{3}{2}}\sec\eta = 1+(\alpha-\varepsilon)P_2+\beta P_4+\sum_2^\infty (v_n+\tfrac{3}{2}u_n)$$
$$+\frac{\varepsilon}{35}\left(\frac{\varepsilon}{6}-\alpha\right)(7+10P_2+18P_4)+\frac{4\varepsilon^2}{105}(7+5P_2-12P_4)$$
$$=1+\frac{\varepsilon}{5}\left(\frac{\varepsilon}{6}-\alpha\right)+\frac{4\varepsilon^2}{15}+P_2\left\{\alpha-\varepsilon+\frac{2}{7}\varepsilon\left(\frac{\varepsilon}{6}-\alpha\right)+\frac{4\varepsilon^2}{21}\right\}$$
$$+P_4\left\{\beta+\frac{18}{35}\varepsilon\left(\frac{\varepsilon}{6}-\alpha\right)-\frac{16\varepsilon^2}{35}\right\}+\sum_2^\infty (v_n+\tfrac{3}{2}u_n)$$
$$=1+\frac{\varepsilon^2}{10}-\frac{\varepsilon\alpha}{5}+P_2\left\{\alpha-\varepsilon+\frac{5\varepsilon^2}{21}-\frac{2\varepsilon\alpha}{7}\right\}$$
$$+P_4\left\{\beta-\frac{13}{35}\varepsilon^2-\frac{18}{35}\varepsilon\alpha\right\}+\sum_2^\infty (v_n+\tfrac{3}{2}u_n)$$
$$=1+\lambda+P_2\{\alpha-\varepsilon+5\mu\}+9\nu P_4+\sum_2^\infty (v_n+\tfrac{3}{2}u_n), \qquad (9.9)$$

in which

$$\lambda=\frac{\varepsilon^2}{10}-\frac{\varepsilon\alpha}{5}, \qquad \mu=\frac{\varepsilon^2}{21}-\frac{2\varepsilon\alpha}{35}, \qquad 9\nu=\beta-\frac{13\varepsilon^2}{35}-\frac{18}{35}\varepsilon\alpha. \quad . \qquad (9.10)$$

N.B.—Different meanings of λ, μ are introduced in (12.12).

Express the harmonic u_n with respect to the pole P′ from which ψ is measured. Then, denoting the value of u_n at P′ by u'_n

$$u_n = u'_n P_n(\mu)+\sum_1^n (a_k\cos k\chi+b_k\sin k\chi),$$

where

$$a_k = A_k\sin^k\psi\,\frac{d^kP_n(\mu)}{d\mu^k}, \qquad b_k=B_k\sin^k\psi\,\frac{d^kP_n(\mu)}{d\mu^k}.$$

A_k, B_k being constants and $\mu=\cos\psi$. Hence

$$\int\frac{u_n}{2\sin\frac{1}{2}\psi}d\omega=\int_0^\pi\int_0^{2\pi}\frac{u_n}{2\sin\frac{1}{2}\psi}\sin\psi\,d\psi\,d\chi=\int_0^\pi\int_0^{2\pi}u_n\cos\tfrac{1}{2}\psi\,d\psi\,d\chi$$
$$=2\pi u'_n\int_0^\pi P_n(\mu)\cos\tfrac{1}{2}\psi\,d\psi, \quad \text{since} \quad \int_0^{2\pi}u_n\,d\chi=2\pi u'_nP_n(\mu)$$
$$=4\pi u'_n\int_0^1 P_n(\mu)\,dx \qquad \text{where} \qquad x=\sin\tfrac{1}{2}\psi.$$

Now $1-2\mu h+h^2=(1-h)^2+4hx^2$ and

$$\int\frac{dx}{\sqrt{1-2\mu h+h^2}}=\Sigma\,h^n\int P_n(\mu)\,dx.$$

So

$$\Sigma\, h^n \int_0^1 P_n(\mu)\, dx = \frac{1}{2\sqrt{h}} \int_0^1 \frac{dx}{\sqrt{a^2+x^2}} \quad \text{where} \quad a^2 = \frac{(1-h)^2}{4h} \quad \text{and} \quad a^2+1 = \frac{(1+h)^2}{4h}$$

$$= \frac{1}{2\sqrt{h}} [\log (2x + 2\sqrt{a^2+x^2})]_0^1 = \frac{1}{2\sqrt{h}} \log \frac{1+\sqrt{a^2+1}}{a}$$

$$= \frac{1}{2\sqrt{h}} \log \frac{1+h+2\sqrt{h}}{1-h}$$

$$= \frac{1}{2\sqrt{h}} \log \frac{1+\sqrt{h}}{1-\sqrt{h}} = \Sigma \frac{h^n}{2n+1}.$$

Hence

$$\int_0^1 P_n(\mu)\, dx = \frac{1}{2n+1},$$

and so

$$\int_0^1 \frac{u_n}{2\sin\frac{1}{2}\psi}\, dx = \frac{4\pi u'_n}{2n+1}.$$

Applying this to (9.9)

$$\frac{1}{4\pi} \int \frac{(1+v)(1+u)^{\frac{3}{2}} \sec\eta}{2\sin\frac{1}{2}\psi}\, d\omega = 1 + \lambda + P'_2\left(\frac{\alpha-\varepsilon}{5} + \mu\right)$$

$$+ \nu P'_4 + \sum_2^\infty \frac{2v'_n + 3u'_n}{2(2n+1)} \qquad (9.11)$$

$$= 1 + \lambda - \frac{7\nu}{18} + P'_2\left(\frac{\alpha-\varepsilon}{5} + \mu - \frac{5}{9}\nu\right)$$

$$+ \frac{35}{18}\nu P'^2_2 + \sum_2^\infty \frac{2v'_n + 3u'_n}{2(2n+1)}. \qquad (9.12)$$

Then

$$(1+u')^{-\frac{1}{2}} = 1 - \tfrac{1}{2}u' + \tfrac{3}{8}u'^2 = 1 + \tfrac{1}{3}\varepsilon P'_2 + \tfrac{1}{6}\varepsilon^2 P'^2_2 - \tfrac{1}{2}\sum_2^\infty u'_n, \qquad (9.13)$$

and

$$\frac{1}{4\pi\sqrt{1+u'}} \int \frac{(1+v)(1+u)^{\frac{3}{2}} \sec\eta}{2\sin\frac{1}{2}\psi}\, d\omega$$

$$= 1 + \lambda - \frac{7\nu}{18} + P'_2\left(\frac{\alpha-\varepsilon}{5} + \frac{\varepsilon}{3} + \mu - \frac{5}{9}\nu\right)$$

$$+ P'^2_2 \left\{\frac{35}{18}\nu + \frac{1}{15}\varepsilon(\alpha-\varepsilon) + \tfrac{1}{6}\varepsilon^2\right\} + \sum_2^\infty \frac{v'_n - (n-1)u'_n}{2n+1}$$

$$= 1 + \lambda - \frac{7\nu}{18} + P'_2\left(\frac{\alpha}{5} + \frac{2\varepsilon}{15} + \mu - \frac{5}{9}\nu\right)$$

$$+ P'^2_2\left(\frac{\varepsilon\alpha}{15} + \frac{\varepsilon^2}{10} + \frac{35}{18}\nu\right) + \sum_2^\infty \frac{v'_n - (n-1)u'_n}{2(n+1)}. \qquad (9.14)$$

10—In deducing the second term of the integral in (9·7) we note that $(\Delta u)^2$ is a second order quantity and so only the principal term in u need be used; also the product $v(\Delta u)^2$ may be neglected. So we must consider the value of

$$I = \int \frac{(\Delta u)^2}{16 \sin^3 \frac{1}{2}\psi}\, d\omega = \tfrac{1}{16}\int_0^{\pi} \frac{\sin\psi\, d\psi}{\sin^3 \frac{1}{2}\psi} \int_0^{2\pi} (\Delta u)^2\, d\chi. \quad . \quad . \quad . \qquad (10.1)$$

Now

$$\Delta u = u - u' = -\tfrac{2}{3}\varepsilon\,(P_2 - P'_2) = \varepsilon\,(\cos^2\theta' - \cos^2\theta), \quad . \quad . \quad . \qquad (10.2)$$

and

$$\cos\theta = \cos\theta' \cos\psi + \sin\theta' \sin\psi \cos\chi,$$

so

$$\Delta u/\varepsilon = \cos^2\theta' \sin^2\psi - \tfrac{1}{2}\sin^2\theta \sin^2\psi\,(1 + \cos 2\chi) - \tfrac{1}{2}\sin 2\theta' \sin 2\psi \cos\chi$$
$$= l + m\cos 2\chi + n\cos\chi. \qquad (10.3)$$

Now

$$\int_0^{2\pi} d\chi = 2\pi \qquad \int_0^{2\pi} \cos m\chi\, d\chi = 0 \qquad \int_0^{2\pi} \cos^2 m\chi\, d\chi = \pi,$$

So

$$\left.\begin{aligned} &\int (\Delta u)^2\, d\chi = \pi\varepsilon^2\,(2l^2 + m^2 + n^2) = A\sin^4\psi + B\sin^2 2\psi \\ &\text{where} \quad A/\pi\varepsilon^2 = 2\,(\cos^2\theta' - \tfrac{1}{2}\sin^2\theta')^2 + \tfrac{1}{4}\sin^4\theta' \\ &\qquad\quad\; B/\pi\varepsilon^2 = \sin^2\theta' \cos^2\theta' \end{aligned}\right\}. \qquad (10.4)$$

Put

$$x = \sin\tfrac{1}{2}\psi, \quad \sin\psi\, d\psi = 4x\, dx, \quad \sin^2\psi = 4x^2\,(1 - x^2), \quad \cos^2\psi = 1 - 4x^2 + 4x^4. \qquad (10.5)$$

then

$$I = \tfrac{1}{16}\int_0^1 \frac{4x\, dx}{x^3}\{16x^4\,(1 - 2x^2 + x^4)\,A + 16x^2\,(1 - 5x^2 + 8x^4 - 4x^6)\,B\}$$

$$= 4\int_0^1 dx\,\{(x^2 - 2x^4 + x^6)\,A + (1 - 5x^2 + 8x^4 - 4x^6)\,B\}$$

$$= 4\left\{A\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) + B\left(1 - \frac{5}{3} + \frac{8}{5} - \frac{4}{7}\right)\right\} = \frac{8}{105}\,(4A + 19B)$$

$$= \frac{8\pi\varepsilon^2}{105}\{18p'^4 - 12p'^2 + 2 + p'^4 - 2p'^2 + 1 - 19p'^4 + 19p'^2\} \quad \text{using (10.4) and with } p' = \cos\theta'$$

$$= \frac{8\pi\varepsilon^2}{105}\,(5p'^2 + 3) = \frac{16}{9}\pi\varepsilon^2\left(\frac{1}{7}P_2' + \frac{1}{5}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10.6)$$

11—We can now write (7.4) as follows, making use of (8.4), (8.8), (9.7), (9.14)

$$V_c/Ga = \tfrac{1}{3}m'\left\{1 - \left(1 + \frac{4}{3}\varepsilon\right)P'_2 + \frac{4}{3}\varepsilon P'^2_2\right\} + \frac{2m'}{3}\left(1 + \frac{4}{15}\varepsilon P'_2\right)$$
$$+ \sum_2^{\infty}\frac{v'_n - (n-1)\,u'_n}{2n+1} + 1 + \lambda - \frac{7\nu}{18} + \left(\frac{\alpha}{5} + \frac{2\varepsilon}{15} + \mu - \frac{5}{9}\nu\right)P'_2$$
$$+ \left(\frac{\varepsilon\alpha}{15} + \frac{\varepsilon^2}{10} + \frac{35}{18}\nu\right)P'^2_2 - \frac{4\varepsilon^2}{9}\left(\frac{1}{5} + \frac{1}{7}P'_2\right) \quad . \qquad (11.1)$$

As this is constant over S, the coefficients of the several harmonics must each vanish, so that

$$v'_n = (n-1)\, u'_n, \qquad \text{(11.2)}$$

and the coefficients of P'_2 and $P_2^{'2}$ must also vanish. So

$$\left.\begin{aligned} -m'\left(\frac{1}{3}+\frac{4}{15}\varepsilon\right)+\frac{\alpha}{5}+\frac{2\varepsilon}{15}+\mu-\frac{5}{9}\nu-\frac{4}{63}\varepsilon^2=0 \\ \frac{4}{9}m'\varepsilon+\frac{\varepsilon\alpha}{15}+\frac{\varepsilon^2}{10}+\frac{35}{18}\nu=0 \end{aligned}\right\} \qquad \text{(11.3)}$$

The approximate value of α to the first order is, from (11.3)

$$\alpha \doteqdot \tfrac{1}{3}\,(5m'-2\varepsilon). \qquad \text{(11.4)}$$

Substituting for α the second equation of (11.3) becomes

$$\frac{5}{9}\varepsilon m'+\frac{1}{18}\varepsilon^2+\frac{35}{18}\nu=0. \qquad \text{(11.5)}$$

Also from (9.10)

$$\left.\begin{aligned} 5\lambda &= \frac{\varepsilon^2}{2}-\varepsilon\alpha=\frac{\varepsilon^2}{2}-\frac{5}{3}\varepsilon m'+\frac{2}{3}\varepsilon^2=\frac{7}{6}\varepsilon^2-\frac{5}{3}\varepsilon m' \\ \mu &= \frac{\varepsilon^2}{21}-\frac{2\varepsilon}{35}\cdot\frac{1}{3}(5m'-2\varepsilon)=\frac{3}{35}\varepsilon^2-\frac{2}{21}\varepsilon m' \\ \beta-9\nu &= \frac{13\varepsilon^2}{35}+\frac{18}{35}\varepsilon\alpha=\frac{1}{35}\{13\varepsilon^2+6\varepsilon(5m'-2\varepsilon)\}=\frac{1}{35}(\varepsilon^2+30\varepsilon m') \end{aligned}\right\} . \qquad \text{(11.6)}$$

Adding (11.5) multiplied by 2/7 to (11.3) and so eliminating ν and substituting for μ

$$-\frac{m'}{3}-\frac{4}{15}\varepsilon m'+\frac{\alpha}{5}+\frac{2\varepsilon}{15}+\frac{3}{35}\varepsilon^2-\frac{2}{21}\varepsilon m'-\frac{4}{63}\varepsilon^2+\frac{10}{63}\varepsilon m'+\frac{1}{63}\varepsilon^2=0,$$

that is

$$\frac{\alpha}{5}-\frac{m'}{3}+\frac{2\varepsilon}{15}+\varepsilon m'\left(-\frac{4}{15}-\frac{2}{21}+\frac{10}{63}\right)+\varepsilon^2\left(\frac{3}{35}-\frac{4}{63}+\frac{1}{63}\right)=0$$

$$\alpha=\tfrac{1}{3}(5m'-2\varepsilon)+\frac{64}{63}\varepsilon m'-\frac{4}{21}\varepsilon^2. \qquad \text{(11.7)}$$

From (11.6) using (11.5)

$$\beta=9\nu+\frac{\varepsilon^2}{35}+\frac{6}{7}\varepsilon m'=-\frac{9}{35}\varepsilon^2-\frac{18}{7}\varepsilon m'+\frac{1}{35}\varepsilon^2+\frac{6}{7}\varepsilon m'$$

$$=-\frac{4}{35}(15\varepsilon m'+2\varepsilon^2). \qquad \text{(11.8)}$$

12—We have accordingly found an expression for gravity in the form

$$g_0 = \mathrm{G}\left(1 + \alpha \mathrm{P}_2 + \beta \mathrm{P}_4 + \sum_2^\infty (n-1)\, u_n\right), \quad \ldots \ldots \quad (12.1)$$

at the surface of the bounding level surface of attraction and rotation (compensated geoid) defined by (*vide* (6.1), (6.3), (6.4))

$$r = a\left(1 - \tfrac{2}{3}\varepsilon \mathrm{P}_2 + \sum_2^\infty u_n\right) = a'\left((1 - \varepsilon' \cos^2\theta + \sum_2^\infty u_n\right), \quad \ldots \quad (12.2)$$

where

$$a' = \text{spheroidal equatorial radius} = a\left(1 + \frac{\varepsilon}{3}\right). \quad \ldots \quad (12.3)$$

Geoidal equatorial gravity is

$$\mathrm{G}'' = \mathrm{G}\left(1 - \frac{\alpha}{2} + \frac{3}{8}\beta + \sum_2^\infty (n-1)\, u''_n\right). \quad \ldots \quad (12.4)$$

in which u''_n is the value of u_n at the equator, where $\theta = 90°$.

Spheroidal equatorial gravity (*vide* (7·2)) is

$$\mathrm{G}' = \mathrm{G}\left(1 - \frac{\alpha}{2} + \frac{3}{8}\beta\right). \quad \ldots \ldots \quad (12.5)$$

Since $m' = a/\mathrm{G}$ and $m = a'/\mathrm{G}'$, and to first order

$$\mathrm{G}' \doteqdot \mathrm{G}'' \doteqdot \mathrm{G}\left(1 - \frac{\alpha}{2}\right), \quad \ldots \ldots \quad (12.6)$$

it follows that

$$m'/m = a\mathrm{G}'/(a'\mathrm{G}) = \left(1 - \frac{\varepsilon}{3}\right)\left(1 - \frac{\alpha}{2}\right) = 1 - \frac{5}{6} m' \text{ by } (11.7).$$

So

and

$$\left.\begin{aligned} m &= m'(1 + \tfrac{5}{6}m') \\ m' &= m(1 - \tfrac{5}{6}m) \end{aligned}\right\}. \quad \ldots \ldots \quad (12.7)$$

Putting p for $\cos\theta$ in (12.1)

$$\begin{aligned} g_0 &= \mathrm{G}\left\{1 + \frac{\alpha}{2}(3p^2 - 1) + \frac{\beta}{8}(35p^4 - 30p^2 + 3) + \sum_2^\infty (n-1)\, u_n\right\} \\ &= \mathrm{G}\left\{1 - \frac{\alpha}{2} + \frac{3\beta}{8} + p^2\left(\frac{3\alpha}{2} + \frac{5\beta}{8}\right) - \frac{35}{8}\beta p^2(1 - p^2) + \sum_2^\infty (n-1)\, u_n\right\}. \end{aligned} \quad (12.8)$$

Using (11.7) and (11.8)

$$\frac{3\alpha}{2} + \frac{5\beta}{8} = \frac{5}{2}m' - \varepsilon + \frac{32}{21}\varepsilon m' - \frac{2}{7}\varepsilon^2 - \frac{15}{14}\varepsilon m' - \frac{1}{7}\varepsilon^2 = \frac{5}{2}m' - \varepsilon + \frac{19}{42}\varepsilon m' - \frac{3}{7}\varepsilon^2,$$

and

$$G\left(\frac{3\alpha}{2}+\frac{5\beta}{8}\right) \doteq G''\left(1+\frac{\alpha}{2}\right)\left\{m\left(\frac{5}{2}+\frac{19}{42}\varepsilon\right)\left(1-\frac{\alpha}{2}-\frac{1}{3}\varepsilon\right)-\varepsilon-\frac{3}{7}\varepsilon^2\right\}$$
$$= G''\left\{m\left(\frac{5}{2}-\frac{8}{21}\varepsilon\right)-\left(\varepsilon+\frac{3}{7}\varepsilon^2\right)\left(1+\frac{5m}{6}-\frac{\varepsilon}{3}\right)\right\}$$
$$= G''\left\{\frac{5}{2}m-\varepsilon-\frac{17}{14}m\varepsilon-\frac{2}{21}\varepsilon^2\right\}. \quad \ldots \quad (12.9)$$

Hence from (12.8)

$$g_0 = G'\left(1+\lambda\cos^2\theta+\mu\sin^2\theta\cos^2\theta+\sum_2^\infty(n-1)u_n\right) \quad \ldots \quad (12.10)$$
$$= G''\left(1+\lambda\cos^2\theta+\mu\sin^2\theta\cos^2\theta+\sum_2^\infty(n-1)(u_n-u''_n)\right), \quad \ldots \quad (12.11)$$

in which

$$\left.\begin{aligned}\lambda &= \frac{5}{2}m-\varepsilon-\frac{17}{14}m\varepsilon-\frac{2}{21}\varepsilon^2\\ \mu &= \frac{15}{2}\varepsilon m+\varepsilon^2\end{aligned}\right\}. \quad (12.12)$$

This may be expressed in terms of the latitude $\phi \doteq 90^\circ - \theta + \varepsilon \sin 2\theta$ so that

$$\theta = 90^\circ - \phi + \varepsilon\sin 2\phi, \quad \ldots \quad (12.13)$$

as follows

$$g_0 = G'\left\{1+\lambda\sin^2\phi+\left(\frac{\mu}{4}-\lambda\varepsilon\right)\sin^2 2\phi+\sum_2^\infty(n-1)u_n\right\} \quad \ldots \quad (12.14)$$
$$= G''\left\{1+\lambda\sin^2\phi+\left(\frac{\mu}{4}-\lambda\varepsilon\right)\sin^2 2\phi+\sum_2^\infty(n-1)(u_n-u''_n)\right\}. \quad (12.15)$$

13—To relate G, G′, G″ to the mass of the earth, it is to be noticed from (2.3) that the mass which is equivalent to the actual distribution τ of density in the earth (or rather within the compensated geoid) is

$$E = -\frac{1}{4\pi}\int\frac{dV}{dn}\,d\sigma+\frac{1}{4\pi}\int\nabla^2U\,dw, \quad \ldots \quad (13.1)$$

E being the total mass of the earth, compensated or not. Using (3.1) and putting $dV/dn = -g_0$

$$E = \frac{1}{4\pi}\int g_0\,d\sigma+\frac{\omega^2}{2\pi}\int dw, \quad \ldots \quad (13.2)$$

the integrations being over the surface S and throughout S respectively. Clearly

$$\int dw = \frac{4\pi}{3}a^3, \quad \ldots \quad (13.3)$$

and using (7.3)

$$\frac{\omega^2}{2\pi}\int dw = \tfrac{2}{3}\,\omega^2 a^3 = \tfrac{2}{3}\,\mathrm{G}a^2 m'. \qquad (13.4)$$

Next

$$\begin{aligned}\int g_0\, d\sigma &= \mathrm{G}a^2\int (1+v)(1+u)^2 \sec\eta\, d\omega\\ &= \mathrm{G}a^2\int\left\{1+\alpha\mathrm{P}_2+\beta\mathrm{P}_4+\sum_2^\infty (n-1)u_n\right\}\\ &\qquad\times\left(1-\frac{4}{3}\varepsilon\mathrm{P}_2+\frac{4}{9}\varepsilon^2\mathrm{P}_2{}^2+2\,\Sigma u_n\right)(1+\tfrac{1}{2}\varepsilon^2\sin^2 2\,\theta)\,d\omega\\ &= \mathrm{G}a^2\int\left\{1+\left(\frac{4}{9}\varepsilon^2-\frac{4}{3}\alpha\varepsilon\right)\mathrm{P}_2{}^2+\tfrac{1}{2}\varepsilon^2\cdot\frac{8}{15}\right\}d\omega\end{aligned}$$

remembering that $\int \mathrm{Y}_n\mathrm{Y}_m\, d\omega = 0$ and using (8.3)

$$\begin{aligned}&= 2\pi\mathrm{G}a^2\int_{-1}^{+1}\left\{1+\frac{4}{9}\varepsilon\,(3\varepsilon-5m')\,\mathrm{P}_2{}^2+\frac{4}{15}\varepsilon^2\right\}dp \text{ using (11.7)}\\ &= 4\pi\mathrm{G}a^2\left[1+\frac{8}{15}\varepsilon^2-\frac{4}{9}\varepsilon m'\right]. \qquad (13.5)\end{aligned}$$

Hence from (13.2), (13.4) and (13.5)

$$\mathrm{E} = \mathrm{G}a^2\left(1+\frac{8}{15}\varepsilon^2-\frac{4}{9}\varepsilon m'+\frac{2}{3}m'\right),$$

or

$$\mathrm{G} = \frac{\mathrm{E}}{a^2}\left(1-\frac{2}{3}m'+\frac{4}{9}m'^2-\frac{8}{15}\varepsilon^2+\frac{4}{9}\varepsilon m'\right), \qquad (13.6)$$

and putting $m' = m\,(1-\frac{5}{6}m)$ from (12.7)

$$\mathrm{G} = \frac{\mathrm{E}}{a^2}\left(1-\frac{2}{3}m+m^2+\frac{4}{9}\varepsilon m-\frac{8}{15}\varepsilon^2\right). \qquad (13.7)$$

By using (12.5) with (11.7) and (11.8) we find from (13.7)

$$\mathrm{G}' = \frac{\mathrm{E}}{a^2}\left(1-\frac{3}{2}m+\frac{\varepsilon}{3}+\frac{9}{4}m^2-\frac{13}{14}\varepsilon m-\frac{11}{21}\varepsilon^2\right), \qquad (13.8)$$

and from (12.4) and (12.5) it is clear that

$$\mathrm{G}'' = \mathrm{G}'\left\{1+\sum_2^\infty (n-1)\,u''_n\right\}. \qquad (13.9)$$

THE LEVEL SURFACES EXTERNAL TO THE GEOID AND THEIR ELLIPTICITIES

14—Reverting to (11.1), since the various harmonic terms vanish, the constant potential V_c over the surface is given by

$$V_c/(Ga) = \frac{1}{3}m' + \frac{2}{3}m' + 1 + \lambda - \frac{7}{18}\nu - \frac{4}{45}\varepsilon^2$$

$$= 1 + m' + \frac{1}{10}\varepsilon^2 - \frac{1}{15}(5m' - 2\varepsilon)\varepsilon + \frac{2}{405}(15\varepsilon m' + 2\varepsilon^2)$$

$$+ \frac{13}{810}\varepsilon^2 + \frac{1}{135}(5m' - 2\varepsilon)\varepsilon \text{ using (9.10), (11.7), (11.8)}$$

$$= 1 + m' + \frac{11}{45}\varepsilon^2 - \frac{2}{9}\varepsilon m' \quad \ldots \quad (14.1)$$

From (7.3) $m' = \omega^2 a/G$; so, using (13.6) and putting

$$M = \omega^2 a^3/E \quad \ldots \quad (14.2)$$

$$M/m' = Ga^2/E = 1 - \frac{2}{3}m' + \frac{4}{9}m'^2 - \frac{8}{15}\varepsilon^2 + \frac{4}{9}\varepsilon m'. \quad \ldots \quad (14.3)$$

Hence, to a first approximation

$$m' = M(1 + \tfrac{2}{3}M) \quad \ldots \quad (14.4)$$

and

$$G = \frac{E}{a^2}\left\{1 - \frac{8}{15}\varepsilon^2 - \frac{2}{3}M\left(1 - \frac{2}{3}\varepsilon\right)\right\}. \quad \ldots \quad (14.5)$$

From (14.1)

$$V_c/a = G\left(1 + \frac{11}{45}\varepsilon^2\right) + Gm'\left(1 - \frac{2}{9}\varepsilon\right)$$

$$= \frac{E}{a^2}\left\{1 - \frac{13}{45}\varepsilon^2 - \frac{2}{3}M\left(1 - \frac{2}{3}\varepsilon\right)\right\} + \frac{EM}{a^2}\left(1 - \frac{2}{9}\varepsilon\right)$$

$$V_c = \frac{E}{a}\left\{1 - \frac{13}{45}\varepsilon^2 + \frac{1}{3}M\left(1 + \frac{2}{3}\varepsilon\right)\right\}. \quad \ldots \quad (14.6)$$

15—Considering a, ε as related *variables*, equation (14.6) gives the potential at a family of level surfaces, while (14.5) gives the corresponding value of G. The difference of potentials at surfaces a, $a + \delta a$ is δV_c. By (14.2) $\delta(M/a) = 2M\delta a/a^2$, so

$$-\delta V_c = \frac{E\delta a}{a^2}\left\{1 - \frac{13}{45}\varepsilon^2 - \frac{2}{3}M\left(1 + \frac{2}{3}\varepsilon\right) + \frac{2}{9}\frac{a\,d\varepsilon}{da}\left(\frac{13}{5}\varepsilon - M\right)\right\}. \quad (15.1)$$

At the equator, where $\theta = 90°$ and $P_2 = -\frac{1}{2}$, by (6.3) $r = a(1 + \frac{1}{3}\varepsilon)$; so the separation of the two surfaces there, where the radius vector deviates from the normal by a second-order quantity, is

$$\delta r = \delta a\left(1 + \frac{1}{3}\varepsilon + \frac{1}{3}\frac{a\,d\varepsilon}{da}\right). \quad \ldots \quad (15.2)$$

The force of gravity there is accordingly $-\delta V_c/\delta r$; and it is also given by (12.4). Hence

$$-\frac{\delta V_c}{\delta r} = G\left\{1 - \frac{\alpha}{2} + \frac{3}{8}\beta + \sum_2^\infty (n-1)\, u''_n\right\}. \qquad (15.3)$$

Now, using (11.7), (11.8), and (14.4)

$$\frac{\alpha}{2} - \frac{3}{8}\beta = \frac{5}{6}M\left(1 + \frac{2}{3}M\right) - \frac{1}{3}\varepsilon + \frac{145}{126}\varepsilon M - \frac{1}{105}\varepsilon^2 \qquad (15.4)$$

Substituting in (15.3) from (14.5), (15.1), (15.2), (15.4)

$$\frac{E}{a^2}\left\{1 - \frac{13}{45}\varepsilon^2 - \tfrac{2}{3}M\left(1 + \tfrac{2}{3}\varepsilon\right) + \frac{2}{9}\frac{a\,d\varepsilon}{da}\left(\frac{13}{5}\varepsilon - M\right)\right\}$$

$$= \frac{E}{a^2}\left\{1 + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\frac{a\,d\varepsilon}{da}\right\}\left\{1 - \frac{8}{15}\varepsilon^2 - \tfrac{2}{3}M\left(1 - \tfrac{2}{3}\varepsilon\right)\right\}$$

$$\times\left\{1 + \tfrac{1}{3}\varepsilon - \frac{5}{6}M\left(1 + \tfrac{2}{3}M\right) - \frac{145}{126}\varepsilon M + \frac{1}{105}\varepsilon^2 + \sum_2^\infty (n-1)\, u''_m\right\}, \qquad (15.5)$$

in which u''_n applies to the surface characterized by a.

16—Equation (15.5) is a differential equation relating a, ε, whose solution will introduce a constant, known in terms of a_0, ε_0. We are only concerned with level surfaces up to heights occurring on the actual earth, which share the earth's rotation; so the extreme variation of a is roughly $a/800$. Second order terms then in (15.5) may be neglected, whence

$$\frac{a\,d\varepsilon}{da} + 2\varepsilon = \frac{1}{a}\frac{d}{da}(a^2\varepsilon) \doteqdot \frac{5}{2}M = \frac{5}{2}\frac{\omega^2 a^3}{E}, \qquad (16.1)$$

$$a^2\varepsilon - a_0^2\varepsilon_0 \doteqdot \frac{1}{2}\frac{\omega^2}{E}(a^5 - a_0^5), \qquad (16.2)$$

which is the law of variation of the ellipticities of the family of level surfaces.

Variation of Gravity with Height

17—Equations (14.5) and (12.1) re-interpreted give the general expression for gravity at all the level surfaces, thus

$$g = \frac{E}{a^2}\left\{1 - \frac{8}{15}\varepsilon^2 - \tfrac{2}{3}M\left(1 - \tfrac{2}{3}\varepsilon\right)\right\}\left\{1 + \alpha P_2 + \beta P_4 + \sum_2^\infty (n-1)\, u_n\right\}. \qquad (17.1)$$

Neglecting variations of second-order terms and using (16.1)

$$\frac{dg}{da} = E\frac{d}{da}\left[\frac{1}{a^2}\left\{1 - \tfrac{2}{3}M + \tfrac{1}{3}(5M - 2\varepsilon)P_2\right\}\right]$$

$$= -\frac{2E}{a^3}\left[1 + \tfrac{1}{3}M - \tfrac{4}{3}\varepsilon P_2\right] = -\frac{2G}{a}\left[1 + M - \tfrac{4}{3}\varepsilon P_2\right]. \qquad (17.2)$$

If h is the separation of a level surface from the geoid then hg, being the work done by unit mass moving from this level surface to the geoid, is constant ; so $hg = h'\text{G}$, where $h' = \delta a \cos \eta$, *vide* (9.5) and to a first order $h' = \delta a$. So from (17.2)

$$\frac{dg}{dh} = \frac{g}{\text{G}}\frac{dg}{da} = -\frac{2g}{a}\left(1 + \text{M} - \tfrac{4}{3}\varepsilon \text{P}_2\right) = -\frac{2\text{G}}{a}\left[1 + m + \left(\tfrac{5}{3}m - 2\varepsilon\right)\text{P}_2\right], \quad (17.3)$$

replacing M by m to which it is equal to the required approximation in the last expression. From (17.2) $d^2g/dh^2 = d^2g/da^2 = 6\text{E}/a = 3\text{G}/a^2$. Hence, reverting to notation used up to §13

$$g - g_0 = -\frac{2\text{G}h}{a}\left[1 + m + \left(\tfrac{5}{3}m - 2\varepsilon\right)\text{P}_2 - \tfrac{3}{2}\frac{h}{a}\right], \quad . \quad . \quad . \quad . \quad (17.4)$$

in which g_0, G, m, ε, a now refer to the geoid and g is gravity at height h vertically above the (compensated) geoidal point where gravity is g_0.

The Relation between the Form of the Geoid and the Force of Gravity at its Surface

18—Sufficient observations of g properly distributed over the face of the globe and reduced to the surface of the compensated geoid will enable us to determine the latter's form with precision. The distribution of stations is considered later (§§ 26, 41), and meantime we proceed to put the basic equations (6.1), (6.4), (12.1) into a form adapted to computation. They may be re-written as follows :—

$$r = a\left(1 - \tfrac{2}{3}\varepsilon \text{P}_2 + \sum_{2}^{\infty} u_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.1)$$

$$g_0 = \text{G}\left(1 + \alpha \text{P}_2 + \beta \text{P}_4 + \sum_{2}^{\infty} (n-1)\, u_n\right), \quad . \quad . \quad . \quad . \quad . \quad (18.2)$$

in which α, β are expressed by (11.7), (11.8) in terms of ε and m', the latter quantity involving the angular velocity of the earth's rotation. Very little is known at present of the value of u_n for any values of n, though analysis of gravity observations has been made to estimate some of the lowest harmonics. The data employed are inadequate for such determinations and could probably be equally well satisfied by various harmonics other than those selected ; but there is little reason why, with present facilities, this situation should not be rectified in a decade.

The determination of geoidal form from gravity observations is essentially linked with the alternative method of determination from measurements by triangulation associated with astronomical observations for latitude, longitude, and azimuth. In the computation of triangulation, the modern practice is to use an ellipsoid of revolution as a reference figure and it is obviously desirable that the determinations by gravity and triangulation should be readily intercomparable.

19—An ellipsoid of revolution of small ellipticity ε may be expressed, neglecting cubes and higher powers of the ellipticity, by

$$r_s = a'' \left(1 - \varepsilon \cos^2 \theta - \tfrac{3}{2}\varepsilon^2 \cos^2 \theta \sin^2 \theta\right), \quad \ldots \ldots \quad (19.1)$$

in which a'' is the equatorial semi-axis, differing slightly from a' of (6.3) owing to inclusion of the ε^2-term in (19.1). Expressing this in terms of mean radius, a, and LEGENDRE functions of the angle measured from the polar axis and making use of (8.3)

$$r_s = a \left\{1 - \tfrac{2}{3}\varepsilon \left(1 + \frac{23}{42}\varepsilon\right) P_2 + \frac{12}{35}\varepsilon^2 P_4\right\}. \quad \ldots \ldots \quad (19.2)$$

This is included in the form (18.1) if we consider the second-order terms $-\dfrac{23}{63}\varepsilon^2 P_2$ and $\dfrac{12}{35}\varepsilon^2 P_4$ as forming part of the general harmonics u_2 and u_4. Similarly, if we were concerned with a tri-axial ellipsoid as a reference figure, we should have the term in $\sin^2 \theta \cos 2(\lambda - \lambda_0)$—in which λ is the longitude—and this too is but a portion of the harmonic u_2. Though at present it is perhaps fanciful to think of other harmonics being surely determined, such need not continue to be the case; and it is easy to show how they may be taken into account. Put

$$u_n = v_n + w_n, \quad \ldots \ldots \ldots \ldots \quad (19.3)$$

in which we suppose that v_n* has in some way been determined, while the remainder, w_n, is not known. Also put

$$R = a\left(1 - \tfrac{2}{3}\varepsilon P_2 + \sum_2^\infty v_n\right), \quad \ldots \ldots \ldots \ldots \quad (19.4)$$

$$\gamma = G\left(1 + \alpha P_2 + \beta P_4 + \sum_2^\infty (n-1)\, v_n\right). \quad \ldots \ldots \quad (19.5)$$

These equations form a reference system and

$$r - R = a \sum_2^\infty w_n = \Delta r, \quad \ldots \ldots \ldots \quad (19.6)$$

$$g_0 - \gamma = G \sum_2^\infty (n-1)\, w_n = \Delta g. \quad \ldots \ldots \ldots \quad (19.7)$$

20—In his classical paper of 1849 STOKES indicated how to relate Δr, Δg. It is clear that at any point P′, which is the pole of the LEGENDRE functions arising in this paragraph

$$\int \frac{\Delta g}{G} \sum_2^\infty \nu P_n \, d\omega = 4\pi \sum_2^\infty w'_n \, \nu \cdot \frac{(n-1)}{2n+1}, \quad \ldots \ldots \quad (20.1)$$

* v_n has here no relation to the v_n occurring in (9.8), (11.2).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

in which ν is a numerical quantity. Putting $\nu = (2n+1)/(n-1)$ the R.H.S. becomes $4\pi \sum_2^\infty w'_n$ and so by (19.6) and (20.1)

$$\frac{\Delta r}{a} = \frac{1}{4\pi G} \int \Delta g f \, d\omega, \quad \ldots \ldots \ldots \ldots \quad (20.2)$$

where

$$f = \sum_2^\infty \frac{2n+1}{n-1} P_n. \quad \ldots \ldots \ldots \ldots \quad (20.3)$$

As shown by STOKES in his original paper and later by POINCARÉ and others

$$f = \operatorname{cosec} \tfrac{1}{2}\psi + 1 - 6 \sin \tfrac{1}{2}\psi - 5 \cos \psi - 3 \cos \psi \log \{\sin \tfrac{1}{2}\psi \, (1 + \sin \tfrac{1}{2}\psi)\}. \quad (20.4)$$

The integration of the function in (20.2) is to be taken over the whole solid angle and the quantities Δr, Δg are those of (19.6), (19.7), which may be regarded as anomalies of radius and gravity from the reference system defined by (19.4), (19.5).

21—To express these anomalies with relation to the ordinary spheroid of reference (19.2), put $v_2 = -\frac{23}{63} \varepsilon^2 P_2$, $v_4 = \frac{12}{35} \varepsilon^2 P_4$ and all the remaining $v_n = 0$. Then R in (19.4) becomes r_s of (19.2) and γ becomes γ_s where

$$\gamma_s = G\left\{1 + \left(\alpha - \frac{23}{63}\varepsilon^2\right) P_2 + \left(\beta + \frac{36}{35}\varepsilon^2\right) P_4\right\}. \quad \ldots \ldots \quad (21.1)$$

It is more convenient to express this as a function of spheroidal latitude. Denote by G'_s the value of γ_s at the equator, so that

$$G'_s = G\left(1 - \frac{\alpha}{2} + \frac{3}{8}\beta + \frac{179}{315}\varepsilon^2\right) = G'\left(1 + \frac{179}{315}\varepsilon^2\right), \quad \ldots \quad (21.2)$$

from (12.5). As in (12.10), from (21.1).

$$\gamma_s = G'\left(1 + \lambda \cos^2 \theta + \mu \sin^2 \theta \cos^2 \theta - \frac{23}{63}\varepsilon^2 P_2 + \frac{36}{35}\varepsilon^2 P_4\right),$$

and

$$-\frac{23}{63} P_2 + \frac{36}{35} P_4 = \frac{179}{315} + \frac{2}{21} p^2 + \frac{9}{2} p^2 (p^2 - 1),$$

so

$$\gamma_s = G'_s\left\{1 + \left(\lambda + \frac{2}{21}\varepsilon^2\right)\cos^2 \theta + \left(\mu - \frac{9}{2}\varepsilon^2\right)\cos^2 \theta \sin^2 \theta\right\}, \quad \ldots \quad (21.3)$$

or, expressed in spheroidal latitude, as in (12.15)

$$\gamma_s = G'_s\left\{1 + \left(\lambda + \frac{2}{21}\varepsilon^2\right)\sin^2 \phi + \left(\frac{\mu}{4} - \lambda\varepsilon - \frac{9}{8}\varepsilon^2\right)\sin^2 2\phi\right\}, \quad . \quad (21.4)$$

λ, μ being given in (12.12). So writing in full

$$\gamma_s = G'_s\left\{1 + \left(\frac{5}{2}m - \varepsilon - \frac{17}{14}m\varepsilon\right)\sin^2 \phi + \frac{1}{8}(\varepsilon^2 - 5\varepsilon m)\sin^2 2\phi.\right\} \quad . \quad (21.5)$$

The numerical values of G'_s, a'' and ε adopted by the International Union of Geodesy and Geophysics at Stockholm (1930) as regards G'_s and at Madrid (1925) as regards a'' and ε are

$$\left.\begin{array}{ll} G'_s = 978\cdot049 \text{ gals } (= \text{cm/sec}^2) & \\ a'' = 6\cdot378{,}388 \times 10^8 \text{ cm} & \varepsilon = 1/297 = 0\cdot003{,}367{,}0 \end{array}\right\}, \quad . \quad (21.6)$$

from which, remembering that the angular velocity of the earth is $2\pi/86164\cdot1$, the denominator being the number of solar seconds in a sidereal day, it follows that

$$m = 1/288\cdot361 = 0\cdot003{,}467{,}87. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.7)$$

Substituting in (21.5)

$$\gamma_s = G'_s\,(1 + 0{,}005{,}288{,}5_1 \sin^2\phi - 0\cdot000{,}005{,}8_7 \sin^2 2\phi), \quad . \quad . \quad (21.8)$$

which may be compared with the International formula (Stockholm).

$$\gamma_s = G'_s\,(1 + 0\cdot005{,}288{,}4 \sin^2\phi - 0\cdot000{,}005{,}9 \sin^2 2\phi). \quad . \quad (21.9)$$

22—The value of Δr given in (20.2) may be used for finding the elevation of the compensated geoid above the reference figure (19.4). If the spheroid (19.1) is used as reference figure then in (20.2) we must for Δg substitute values of Δg_s where

$$\Delta g_s = g_0 - \gamma_s, \quad . \quad . \quad . \quad (22.1)$$

and γ_s is given by (21.8). This partly answers the question of locating a unique and universal reference spheroid with regard to a particular survey. But in such a case it is also necessary to know the relative tilts of the compensated geoid and the reference figure, and we proceed to provide for doing this. In (20.2) ψ is the angle between the radius vector through the point P′ at which Δr is to be determined and the point Q where the gravity anomaly Δg occurs. In fig. 2, C is the pole of rotation. If P′Q is in azimuth χ and P′P″ in azimuth A and P′P″ $= \delta s$ then

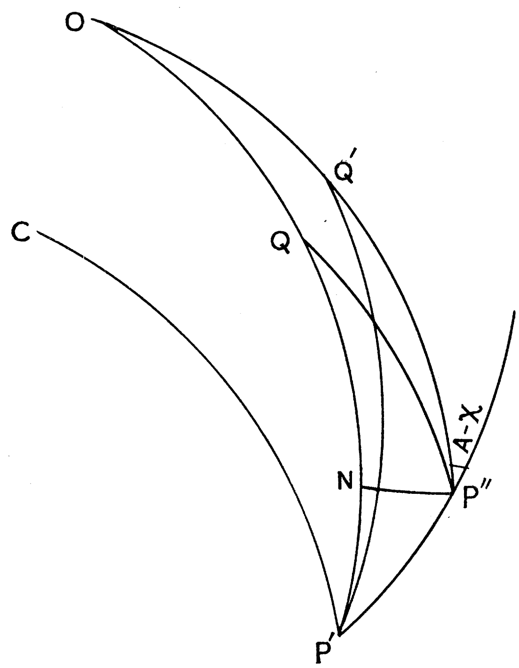

Fig. 2

making P″NQ a right angle, qP′P″ $= \mathrm{A} - \chi$ and

$$-a\delta\psi \doteqdot -r\delta\psi = \mathrm{QP}' - \mathrm{QP}' \doteqdot \mathrm{PN} = \delta s \cos(\mathrm{A} - \chi).$$

Hence

$$\text{tilt} = \mathscr{L}\frac{\delta\,\Delta r}{\delta s} = \frac{a}{4\pi \mathrm{G}}\,\mathscr{L}\int \Delta g \frac{\delta f}{\delta\psi}\cdot\frac{\delta\psi}{\delta s}\,d\omega = -\frac{1}{4\pi\mathrm{G}}\int \Delta g\,\frac{df}{d\psi}\cos(\mathrm{A} - \chi)\,d\omega. \quad (22.2)$$

With notation of (39.3)

$$\int \Delta g \frac{df}{d\psi} \cos \chi \, d\omega = \int_0^\pi \int_0^{2\pi} \Delta g \frac{df}{d\psi} \cos \chi \sin \psi d\psi d\chi = \Sigma B_n \int_0^\pi \frac{df}{d\psi} \sin^2 \psi \frac{dP_n}{dp} d\psi.$$

The factor $\frac{df}{d\psi} \sin^2 \psi$ is finite throughout the range, *vide* (36.6), and dP_n/dp is a polynomial in $\cos \psi$, hence the integral in (22.2) is finite and determinate. The tilt given by (22.2) is the deviation of geoidal vertical from spheroidal vertical, in azimuth A measured from north round by east. The northerly and easterly components of the deviation—displacement of geoidal nadir—are derived by putting $A = 0$ and $A = 90°$ and are (in seconds)

$$\left.\begin{aligned} \eta'' &= -\frac{\operatorname{cosec} 1''}{4\pi G} \int \Delta g_s \frac{df}{d\psi} \cos \chi \, d\omega \\ \xi'' &= -\frac{\operatorname{cosec} 1''}{4\pi G} \int \Delta g_s \frac{df}{d\psi} \sin \chi \, d\omega \end{aligned}\right\}, \tag{22.3}$$

the integrations being taken throughout the solid angle and $d\omega$ being $\sin \psi d\psi d\chi$.

The above formulæ are suitable for the case when values of Δg at widely separated points are given. It will be seen later (§ 38) that $df/d\psi \sin \psi$ becomes infinite when $\psi = 0$, which creates difficulties in the evaluation of the integral by quadratures. For this case we may use a modified formula. Consider two adjacent points P′, P″, the angular value of P′, P″ being $\delta\sigma$. Draw great circles through P′, P″, each making the same angle with P′, P″, and suppose that these meet in O. Then sin OP′ = sin OP″ so that OP′ + OP″ = π, OP′ − OP″ = $\delta\sigma \cos \overline{\chi - A}$, whence

$$OP' = \frac{\pi}{2} + \frac{\delta\sigma}{2} \cos \overline{\chi - A}. \quad \ldots \ldots \ldots \ldots \tag{22.4}$$

On OP′ and OP″ take points Q, Q′ such that P′ Q = P″ Q′ = ψ. Then by (20.2)

$$\frac{\Delta r}{a} = \frac{1}{4\pi G} \int \Delta g f \, d\omega, \qquad \frac{\Delta r'}{a} = \frac{1}{4\pi G} \int \Delta g' f \, d\omega,$$

where Δg, $\Delta g'$ apply to Q, Q′ respectively. Hence

$$-\frac{\Delta r' - \Delta r}{a\delta\sigma} = -\frac{1}{4\pi G} \int \frac{\Delta g' - \Delta g}{\delta\sigma} f \, d\omega. \quad \ldots \ldots \tag{22.5}$$

In the limit, when $\delta\sigma = 0$, this is the deviation of the geoidal zenith in direction P′ P″. Clearly P′ Q′ − P′ Q = P′ Q′ − P″ Q′ = $\delta\sigma \cos \overline{\chi - A}$, and from triangles P′ Q′ O, P′ Q′ P″

$$Q' P' Q = -\delta\chi = \sin OQ' \operatorname{cosec} OP' \sin P'' Q' P'$$

$$\doteqdot \sin OQ' \sin P' P'' \operatorname{cosec} P' Q' \sin \overline{\chi - A} \doteqdot \delta\sigma \cot \psi \sin \overline{\chi - A}$$

since $OP' \doteqdot \frac{1}{2}\pi$ from (22.4). Hence

$$\Delta g' - \Delta g = \delta\psi \frac{d\Delta g}{d\psi} + \delta\chi \frac{d\Delta g}{d\chi} = \delta\sigma \cos \overline{\chi - A} \frac{d\Delta g}{d\psi}$$
$$- \delta\sigma \cot \psi \sin \overline{\chi - A} \frac{d\Delta g}{d\chi}. \quad (22.6)$$

Hence from (22.5), (22.6)

$$-\frac{dr}{ds} = -\frac{dr}{a\delta\sigma} = -\frac{1}{4\pi G}\int \mathscr{L} \frac{\Delta g' - \Delta g}{\delta\sigma} f\, d\omega$$
$$= -\frac{1}{4\pi G}\iint \left(\cos \overline{\chi - A} \frac{d\Delta g}{d\psi} - \cot \psi \sin \overline{\chi - A} \frac{d\Delta g}{d\chi}\right) f \sin \psi \, d\psi \, d\chi. \quad (22.7)$$

If we put

$$u = \sin \psi \sin \chi, \; v = \sin \psi \cos \chi \quad \ldots\ldots\ldots \quad (22.8)$$

$$\frac{d}{d\psi} = \frac{du}{d\psi}\frac{\partial}{\partial u} + \frac{dv}{d\psi}\cdot\frac{\partial}{\partial v} = \cos \psi \left(\sin \chi \frac{\partial}{\partial u} + \cos \chi \frac{\partial}{\partial v}\right)$$

$$\frac{d}{d\chi} = \frac{du}{d\chi}\frac{\partial}{\partial u} + \frac{dv}{d\chi}\frac{\partial}{\partial v} = \sin \psi \left(\cos \chi \frac{\partial}{\partial u} - \sin \chi \frac{\partial}{\partial v}\right)$$

$$\cos \overline{\chi - A} \frac{\partial}{\partial \psi} - \cot \psi \sin \overline{\chi - A} \frac{\partial}{\partial \chi} = \cos \psi \left\{\sin A \frac{\partial}{\partial u} + \cos A \frac{\partial}{\partial v}\right\}$$

$$-\frac{d\Delta r}{ds} = -\frac{1}{4\pi G}\iint \left(\sin A \frac{\partial \Delta g}{\partial u} + \cos A \frac{\partial \Delta g}{\partial v}\right) f \sin \psi \cos \psi \, d\psi \, d\chi. \quad (22.9)$$

Hence

$$\left.\begin{aligned} \xi'' &= \left(-\frac{dr}{ds}\right)_{A=\frac{1}{2}\pi} \operatorname{cosec} 1'' = -\frac{\operatorname{cosec} 1''}{4\pi G}\iint \frac{\partial \Delta g}{\partial u} f \sin \psi \cos \psi \, d\psi \, d\chi \\ \eta'' &= \left(-\frac{dr}{ds}\right)_{A=0} \operatorname{cosec} 1'' = -\frac{\operatorname{cosec} 1''}{4\pi G}\iint \frac{\partial \Delta g}{\partial v} f \sin \psi \cos \psi \, d\psi \, d\chi \end{aligned}\right\}. \quad (22.10)$$

Put $u = x/a$, $v = y/a$, where a is the earth's radius in miles; then we replace $\partial \Delta g/\partial u$ by $a\, \partial \Delta g/\partial x$ and $\partial \Delta g/\partial v$ by $a\, \partial \Delta g/\partial y$ in which for reasonable distances from the origin $\partial \Delta g/\partial x$, $\partial \Delta g/\partial y$ are gradients of Δg per mile, approximately. For greater distances the difference between $\sin \psi/\psi$ and unity would need to be considered.

In (22.10) nothing has been said about the limits of the integration, which have been tacitly assumed to be over the whole solid angle. It is, however, the intention to limit the region of integration and this will give rise to a sudden change from the value of Δg at Q just inside the boundary, to zero at Q′ just outside it. The boundary effect occurs between ρ and $\rho + \Delta\rho$ and $\Delta\rho = \Delta x \sin \chi$. This contributes

$$-\frac{\operatorname{cosec} 1''}{4\pi G}\int_{\chi_1}^{\chi_2}\left[\frac{\Delta g}{\Delta x} f \sin \psi \; \Delta r\right] d\chi = -\frac{\operatorname{cosec} 1''}{4\pi G}\int_{\chi_1}^{\chi_2} \Delta g\, f \sin \psi \sin \chi \, d\chi,$$

the integration being all round the boundary.

Finally the deflection due to a limited region is

$$\delta\xi'' = -\frac{\text{cosec } 1''}{4\pi G}\left\{\int_{\chi_1}^{\chi_2}\int_{\rho_1}^{\rho_2}\frac{\partial\,\Delta g}{\partial x}f\sin\psi\,d\rho\,d\chi - \int_{\chi_1}^{\chi_2}\Delta g\,f\sin\psi\sin\chi\,d\chi\right\}. \quad (22.11)$$

with a similar expression for $\delta\eta''$ in which x is changed to y and $\sin\chi$ to $\cos\chi$.

Ordinarily the limited region will be bounded by a circle of radius R, centred at P′, for which case

$$\delta\xi'' = -\frac{\text{cosec } 1''}{4\pi G}\left\{\int_0^{2\pi}\int_0^{R}\frac{\partial\,\Delta g}{\partial x}\,f\sin\psi\,d\rho\,d\chi - \int_0^{2\pi}(\Delta g f\sin\psi)_R\sin\chi\,d\chi\right\}, \quad (22.12)$$

with a similar expression with y in place of x and $\cos\chi$ for $\sin\chi$ for $\delta\eta''$.

It is throughout to be remembered that the equations herein deduced relate to the compensated geoid. The difference between this and the actual geoid can be easily calculated from suitable maps showing the topography; and of course the bases of this reduction should be the same as those employed in reducing the observed gravity results to compensated geoidal level.

The Astronomical Potential

23—The attractional potential of the earth at distance R, greater than a' (*e.g.*, at the moon) is easily found from (7.4) for

$$V'' = V_c - U = W + X \quad \dots\dots\dots \quad (23.1)$$

in which now R′ is measured from any external point, and is thereby seen to be

$$V'' = \frac{E}{R}\left[1 + \lambda''\left(\frac{a}{R}\right)^2 P_2 + \mu''\left(\frac{a}{R}\right)^4 P_4 + \Sigma f(n)\left(\frac{a}{R}\right)^n u_n\right]. \quad . \quad (23.1)$$

The coefficients can be found by direct integration, but more simply by equating the surface value of V″ to $V_c - U_c$. Thus, making use of (14·3) and (8·4)

$$V_c - U_c = \text{constant} - \frac{1}{3}m'\frac{E}{a}\left(1 - \frac{2}{3}m'\right)\left\{1 - \left(1 + \frac{4}{3}\varepsilon\right)P_2 + \frac{4}{3}\varepsilon\,P_2{}^2\right\}$$

$$= \frac{E}{a}\left[1 + \frac{2}{3}\varepsilon - \Sigma u_n + \frac{4}{9}\varepsilon^2 P_2 + \lambda''(1 + 2\varepsilon P_2)P_2 + \mu'' P_2 + \Sigma f(n)\,u_n\right] \quad . \quad (23.3)$$

the R.H.S. being found from (23·2) and (7·5).

Comparing coefficients of like harmonics

$$f(n) = 1 \quad \dots\dots\dots\dots \quad (23.4)$$

and expressing P_4 in terms of P_2 and $P_2{}^2$, and comparing coefficients of these latter quantities

$$\left.\begin{aligned} m'\left(1 - \frac{2}{3}m'\right)\left(1 + \frac{4}{3}\varepsilon\right) &= \frac{2}{3}\varepsilon + \lambda'' - \frac{5}{9}\mu'' \\ -\frac{4}{9}\varepsilon m' &= \frac{4}{9}\varepsilon^2 + 2\lambda''\varepsilon + \frac{35}{18}\mu'' \end{aligned}\right\} \quad \dots\dots \quad (23.5)$$

whence, after using (12·7) to introduce m in place of m' we find

$$\left.\begin{aligned} \lambda'' &= \frac{m-2\varepsilon}{3} - \frac{1}{2}m^2 + \frac{8}{63}m\varepsilon + \frac{16}{63}\varepsilon^2 = 0\cdot001{,}094{,}50 \\ \mu'' &= \frac{16}{35}\varepsilon^2 - \frac{4}{7}\varepsilon m = 0\cdot000{,}001{,}17 \text{ with } \varepsilon = 1/297 \end{aligned}\right\}. \qquad (2.36)$$

To incorporate the international spheroid we proceed as in Sections 21, 22, obtaining

$$V''_{int.} = \frac{E}{R}\left[1 + \left(\lambda'' - \frac{23}{63}\varepsilon^2\right)\left(\frac{a}{R}\right)^2 P_2 + \left(\mu'' + \frac{12}{35}\varepsilon^2\right)\left(\frac{a}{R}\right)^4 P_4 + \Sigma\left(\frac{a}{R}\right)^n w_n\right] \quad (23.7)$$

in which the w's represent departures from the international spheroid.
Owing to the smallness of the factor a/R it is probably sufficient to write

$$V'' = \frac{E}{R}\left\{1 + \tfrac{1}{3}(m-2\varepsilon)\left(\frac{a}{R}\right)^2 P_2(\cos\theta) + \left(\frac{a}{R}\right)^2 w_2\right\}, \quad . \quad . \quad . \quad (23.8)$$

in which w_2 is the excess of the second harmonic over what is taken into account in the adopted figure of reference—*vide* (19.3). As an example w_2 may represent ellipticity of the earth's equator. If confidence is felt in its reality and we accept an estimate made by HEISKANEN we can put $w_2 = 27\cdot10^{-6}\sin^2\theta\cos 2(\lambda-\lambda_0)$, where λ_0 is 18° east longitude, which is a diurnal perturbation.

The difference due to the use of the compensated geoid in place of the geoid is clearly negligible to the precision considered in (23.3).

PRELIMINARY CONSIDERATIONS FOR A WORLD GRAVITY SURVEY

24—In speaking of *Figures of Reference* for the earth we may be more specific in replacing the word " earth " by " compensated geoid," so avoiding the complication of topography. The complete figure is represented in this work by (18.1), in which a whole series of small harmonic terms is as yet unknown, but derivable from an adequate gravimetric survey with the help of (19.6) with (19.4) and (20.2) in which simple quadratures replace the need of harmonic analysis. We may regard (19.4) as defining a figure of reference; and in this we can assign to the several v's such values as observations of any kind appear to warrant. As a particular instance of this we may take the form (19.2) which is an ellipsoid of revolution or what is usually more briefly designated a spheroid. This spheroid has its centre coincident with the centre of gravity of the actual earth and its minor axis parallel to the earth's axis of diurnal rotation; and as the rotation axis must pass through the centre of gravity, the minor axis is actually coincident with the axis of diurnal rotation. This spheroid is accordingly uniquely located with regard to the earth and, if values of ε and a are assigned, it is completely defined. As suggested in § 19 some small (second order) residue of a P_2 term may remain in the harmonic u_2; so that ε may differ from the ideal value by a second-order quantity without in any way invalidating

(19.1) as a figure of reference. With the growth of data new evaluations of ε may be made; but until all the second-order quantities (viz., all the u's or Δr) are properly determined the ideal value of ε cannot be found. Nor, so far as I can see, has this ideal ε any particular significance.

The value of a is not determinable from gravity observations, but can be found from arcs or areas of triangulation. Hitherto the results so derived have been burdened with error due to uncertainty as to deviation of the vertical; and in this respect full gravity results applied to (22.3) should enable a considerable improvement to be made. So far as gravity results and the form of the geoid is concerned, it is of no great consequence that a should be more precisely determined than has already been done. It enters into the determination of the first-order quantity m. It also enters into any determination from gravity measurements of the mean density of the earth, the gravitational constant being supposed independently known. Terrestrial bases, however, such as Greenwich—Cape Town, are employed for lunar parallax observations; and, for fixing the scale of the universe, are required to the greatest attainable accuracy. Here is one reason for a thorough world gravity survey.

Now consider the figure of the earth from the point of view of triangulation and related measurements.

All measures of triangulation are made by instruments levelled (or corrected for level) to the geoidal horizontal; and the results are accordingly geoidal angles. These require reduction (usually small, but on occasion as great as 3″) to the spheroid; and the reduced angles alone can be correctly used for computation of position on and with relation to the spheroid. The corrections are determinate if the deviation of the vertical (*i.e.*, angle between the vertical or normal to the geoid and the normal to the spheroid) is known at each station, a condition at present far from realized. Assuming that this were all satisfactory, it is possible to compute the triangulation quite logically on any reasonable spheroid of reference. Without the reduction, the computation cannot be made to do justice to the observations on *any* spheroid, no matter how carefully selected; for the anomalies between the geoid and selected spheroid are greater than the discrepancies between any two reasonably chosen spheroids, at least to moderate distances from the origins.

The value of geodetic survey will be increased when all its results are referred to one unique reference system. Some progress in this direction is made when two adjoining surveys are linked up; but obviously such connections cannot be made across oceans, nor can most geodetic processes be carried out at sea. The solution of the main problems of geodesy can most easily and practically be effected by gravity survey, which is not now restricted to land areas. We proceed to consider how much labour will be entailed in such a gravity survey, and the general lay-out of the gravity stations that will be required.

25—Although numerous determinations of gravity have been made since 1849, the year of STOKES's paper, and latterly several hundred have been made at sea, still only about 3% of the surface of the globe has been gravimetrically surveyed, in addition to which observations have been made at a considerable number of detached points distributed rather at hazard and according to convenience of access. It is necessary to provide sufficient data for the evaluation of the integrals $\frac{a}{4\pi G}\int \Delta g f \, d\omega$ and $\frac{1}{4\pi G}\int \Delta g \frac{df}{d\psi} \frac{\cos}{\sin} \chi \, d\omega$ taken throughout the solid angle. So g must be known with sufficient detail over the earth's surface to enable these integrals to be evaluated substantially correctly by quadratures. It is convenient to recall that f is given in (20.4) and that $d\omega = \sin\psi \, d\chi \, d\psi$. Values of $f \sin \psi$ are given in Table IV of § 38.

26—An inspection of the integrals and of the values of $f \sin \psi \equiv \phi$ and $df/d\psi \sin \psi$ makes it clear that, for the determination of Δr, ξ, η at any point P′, values of g are required in greater detail in the vicinity of P′ than elsewhere; while elsewhere station density could be varied with economy of observational labour. But it is not only at one point that we require to evaluate the integrals, but at many scattered points, *e.g.*, origins of all surveys. It follows that the practical ends can be met by

(*a*) a general world gravity survey, with stations uniformly spaced over the whole globe;

(*b*) local gravity surveys of suitable station density in the neighbourhood of each point P′.

From (*a*) alone the generalized form of the geoid can be found. For greater detail, at or near a point, a local detail survey, as in (*b*), or some alternative is required. An alternative is that based on measures of deviation of the vertical, sufficiently detailed to allow of the delineation of the geoid with reference to a locally selected reference figure. Such measurements are also required for the rigid reduction of the triangulation of the survey concerned (*vide* § 24), though hitherto this refinement has been neglected. The counsel of perfection is to observe deviations of the vertical at all triangulation stations for proper reduction of the triangulation and *either* similar observations at a sufficient number of intermediate points as will allow contours of equal deviations to be drawn sensibly correctly *or* to observe g at numerous stations with a distribution suited to the evaluation of the integrals expressing geoidal rise and deviation. A combination of these two alternatives would be even better than either alone.

Before we can decide on the proper spacing of gravity stations to meet (*a*), some study of the nature and amount of gravity anomalies is necessary. A certain amount of conjecture must be made, based on results already obtained from relatively small portions of the globe; and any conjecture will be open to review later on in the light of results found in the future. First let us try to fix ideas as to the order of accuracy which is attainable and desirable for our purpose.

27—Triangulation emanates from measured bases and an accuracy of 1 in 10^6 may be taken as representing good modern base-line practice. Good geodetic triangulation may have a probable error of a single adjusted angle ($= 0''\cdot551 \; m$, where $m = (\Sigma \Delta^2/3n)^{\frac{1}{2}}$ is Ferrero's criterion, Δ being triangular error and n the

TABLE I—$v = g - \gamma_c$

ϕ \ λ	72°			74°			76°			78°		
	No.	v	v^2	No.	v	v^2	No.	v	v^2	No.	v	v^2
28°												
	210	8	64	213	25	625	103	−5	25	119	−13	169
	211	12	144	214	−16	256	104	17	289	120	−29	841
	212	8	64	215	62	3844	105	15	225	122	−39	1521
	—	—	—	224	44	1936	106	11	121	263	−26	676
	—	—	—	—	—	—	107	−8	64	—	—	—
	n 3	Σv 28	Σv^2 272	n 4	Σv 115	Σv^2 6661	n 5	Σv 30	Σv^2 724	n 4	Σv − 107	Σv^2 3207
26°												
	114B	47	2209	216	− 4	16	98	19	361	95	− 2	4
	115	29	841	222	3	9	99	39	1521	96	26	676
	117	23	529	223	−21	441	102	29	841	100	14	196
	118	20	400	—	—	—	—	—	—	101	− 7	49
	n 4	Σv 119	Σv^2 3979	n 3	Σv − 22	Σv^2 466	n 3	Σv 87	Σv^2 2723	n 4	Σv 31	Σv^2 925
24°												
	111	− 8	64	47	−11	121	50	5	25	59	11	121
	112	36	1296	48	−15	225	52	30	900	60	4	16
	114B	47	2209	49	−19	361	56	21	441	219	−22	484
	—	—	—	217	13	169	97	22	484	220	−21	441
	—	—	—	—	—	—	218	27	729	—	—	—
	n 3	Σv 75	Σv^2 3569	n 4	Σv − 32	Σv^2 876	n 5	Σv 105	Σv^2 2579	n 4	Σv − 28	Σv^2 1062
22°												
	110	32	1024	53	20	40	51	47	2209	58	38	1444
	113	49	2401	233	−35	1225	52	30	900	67	36	1296
	116	15	225	234	−14	196	54	26	676	205	− 2	4
	—	—	—	—	—	—	55	31	961	237	5	25
	—	—	—	—	—	—	235	−15	225	—	—	—
	—	—	—	—	—	—	236	− 9	81	—	—	—
	n 3	Σv 96	Σv^2 3650	n 3	Σv − 29	Σv^2 1461	n 6	Σv 110	Σv^2 5052	n 4	Σv 77	Σv^2 2769
20°												

number of triangles) ranging from 0″·1 to 0″·3; but for a good deal of existing geodetic triangulation, executed during the last century, this error may average 0″·5. The high accuracy of the base is soon lost in the triangulation. Base-lines were formerly measured only with very great labour and so were less frequently introduced than might be desired. Taking the great systems of existing triangulation and remembering that they are carefully adjusted networks, I think the average linear accuracy may be put somewhere between 1 in 50,000 and 1 in 200,000, the higher precision occurring in the more modern work plentifully endowed with base-lines. To do justice to such results an accuracy of 1 or 2 in 10^6 in the elements of location of the reference figure should be attained; or more specifically the probable error of Δr should be about 10 metres and that of the deviations ξ, η about 0″·3 (*cf.* § 46). The absolute precision of determination of our integrals depends partly on the attainable precision in determination of g. At a land station under and v^2

λ / ϕ	80°			82°			84°			86°
	No.	v	v^2	No.	v	v^2	No.	v	v^2	
28°										
	123	−54	2916	71	−27	729	92	−42	1764	
	125	−62	3844	93	−57	3249	128	−88	7744	
	126	−85	7225	94	−70	4900	129	−99	9801	
	262	−23	529	127	−91	8281	—	—	—	
	—	—	—	—	—	—	—	—	—	
	n 4	Σv −224	Σv^2 14514	n 4	Σv −245	Σv^2 17159	n 3	Σv −229	Σv^2 19309	
26°										
	69	− 3	9	83	2	4	84	25	625	
	70	9	81	89	9	81	86	3	9	
	260	15	225	90	− 5	25	87	−25	625	
	261	13	169	91	−14	196	88	−28	784	
	n 4	Σv 34	Σv^2 484	n 4	Σv − 8	Σv^2 306	n 4	Σv −25	Σv^2 2043	
24°										
	61	7	49	63	8	64	85	30	900	
	62	29	841	64	13	169	252	6	36	
	68	30	900	259	2	4	258	14	196	
	221	6	36	—	—	—	—	—	—	
	—	—	—	—	—	—	—	—	—	
	n 4	Σv 72	Σv^2 1826	n 3	Σv 23	Σv^2 237	n 3	Σv 50	Σv^2 1132	
22°										
	65	− 3	9	247	− 2	4	6	6	36	
	66	− 3	9	248	−24	576	169	27	729	
	238	− 8	64	249	4	16	250	32	1024	
	—	—	—	—	—	—	251	− 7	49	
	—	—	—	—	—	—	—	—	—	
	n 3	Σv −14	Σv^2 82	n 3	Σv −22	Σv^2 596	n 4	Σv 58	Σv^2 1838	
20°										

reasonably good conditions a p.e. of 2 in 10^5 is fairly representative of ordinary practice, while at sea stations the corresponding p.e. may be taken as 3 in 10^6. These figures give a *prima facie* appearance of being of the order required; but closer consideration is necessary.

The Error of Representation of a Single Gravity Station

28—In evaluating our integrals by quadratures we must divide the whole globe into a reasonable number of approximately equal areas; and for each of these we must assume that the anomaly of gravity observed at one point represents the whole of that area. Denote the average discrepancy between the actual reduced value of gravity at a point and the mean of a number of such values over the associated region by E_r, which we call the "total error of representation." This comprises error of observation, e_0, error of reduction to geoidal level, e_1, and simple error of representation, e_r. The latter would be the same as E_r if the two other errors were nil. There may also be a systematic error due to apparatus e_a. Let us now consider the order of magnitude of these errors.

29—To obtain an estimate of E_r and the law of variation of e_r I have considered a group of observations of g made in India. These fall fairly regularly distributed over a belt of 8° of latitude and 14° of longitude, embracing the area of considerable anomaly of the Gangetic plain and also the less disturbed region to the west. How far the results can be regarded as typical of the whole earth will only be known when gravity survey has been much extended over the earth. All these observations have been reduced on the HAYFORD hypothesis of compensation. For other reasons we have decided that our problem requires that results should be reduced on some basis of compensation so as to remove the effect of matter external to the geoid, the compensation being only an artifice used for this purpose and its actuality being of no consequence. The HAYFORD system meets the requirements. It is, however, found that the HAYFORD anomalies are definitely smaller than those of the free-air reduction—in the present case 60%.

Our area is divided into 28 squares of 2 degrees of latitude and 2 degrees of longitude and the total number of stations is 104; so that on the average 3·7 stations occur in each square. By a little justification it is arranged that 11 squares have 3 stations each, 14 have 4, 2 have 5 and 1 has 6 stations; so that without much error each area unit of $(2°)^2$ may be regarded as having the same weight. As a tentative measure the compensated anomalies are regarded as occurring by hazard; and the figures are found to justify this. If v is any anomaly and V is the mean of n anomalies then $\frac{1}{n}\Sigma\,(v-\mathrm{V})^2$ is the mean-square anomaly E^2. Clearly

$$n\mathrm{V} = \Sigma\, v, \quad \ldots \ldots \ldots \ldots \quad (29.1)$$

so

$$\mathrm{E}^2 = \frac{1}{n}\Sigma\,(v-\mathrm{V})^2 = \frac{1}{n}(\Sigma\, v^2 - 2\mathrm{V}\,\Sigma\, v + n\mathrm{V}^2) = \frac{1}{n}\Sigma\, v^2 - \left(\frac{1}{n}\Sigma\, v\right)^2. \quad (29\cdot 2)$$

We accordingly proceed to find $\frac{1}{n}\Sigma v^2$ and $\frac{1}{n}\Sigma\, v$ for each square. The data (taken from Survey of India Geodetic Reports VI, VII, VIII) are exhibited in Table I. In this are given station numbers, v and v^2, grouped by 2° squares; at the end of each of which are given n, the number of stations, Σv and Σv^2. Values of $\frac{1}{n}\Sigma v^2$, $\frac{1}{n}\Sigma v$ are abstracted from Table I and given in Table II, from which are derived values of E^2 for each 2° square given at end of Table II, with the root mean square $\mathrm{E}\,(2, 2) = \sqrt{\Sigma \mathrm{E}^2/28}$ in mg.

Regarding each 2° square as a unit of equal weight, these squares are grouped by the summation of the quantities in Table II for areas $(2x)°$ of latitude and $(2y)°$

TABLE II—$\Delta\phi = 2°$ $\Delta\lambda = 2°$

ϕ \ λ	72°		74°		76°		78°		80°		82°		84°		86°
28°															
	91	9	1665	29	145	6	802	−27	3628	−56	4290	−61	6436	−76	
26°															
	995	30	155	−7	908	29	231	8	121	9	77	−2	511	−6	
24°															
	1190	25	219	−8	516	21	266	−7	457	18	79	8	377	17	
22°															
	1217	32	487	−10	842	18	692	19	27	−5	199	−7	460	15	
20°															

Values of E^2

28°							
	10	824	109	73	492	569	660
26°							
	95	106	67	165	40	73	475
24°							
	565	155	75	217	133	15	88
22°							
	193	387	518	331	2	150	235
20°							
Total	863	1472	769	786	667	807	1458

Grand total 6822.

$$E\,(2,\,2) = \sqrt{6822/28} = \sqrt{244} = 15\cdot 6 \text{ mg}$$

of longitude for various values of x, y. The values of E $(2x, 2y)$ are computed by the help of (29.2) and the results are included in Table III.

TABLE III

$\Delta\phi$	$\Delta\lambda$	x	y	E $(2x, 2y)$	$(x+y)^{\frac{1}{2}}$	$(x^2+y^2)^{\frac{1}{4}}$	$x^{\frac{1}{2}}+y^{\frac{1}{2}}$	$\dfrac{E}{(x+y)^{\frac{1}{2}}}$	$\dfrac{E}{(x^2+y^2)^{\frac{1}{4}}}$	$\dfrac{E}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$
2°	2°	1	1	15·6 mg	1·41	1·19	2·00	11·1	13·1	7·80
4	2	2	1	23·3	1·73	1·49	2·41	13·5H	15·6H	9·66H
4	4	2	2	24·6	2·00	1·68	2·82	12·3	14·6	8·62
8	2	4	1	27·6	2·24	2·03	3·00	12·3	13·6	9·20
8	4	4	2	27·8	2·45	2·11	3·41	11·4	13·2	8·16
8	8	4	4	29·2	2·83	2·38	4·00	10·3	12·3	7·30
2	14	1	7	27·2	2·83	2·66	3·65	9·6	10·2L	7·45
4	14	2	7	29·8	3·00	2·70	4·06	9·9	11·0	7·34
8	14	4	7	31·1	3·32	2·84	4·65	9·4L	11·0	6·69L
							$\dfrac{\text{Range}}{\text{Mean}}$	$\dfrac{4\cdot 1}{11\cdot 1} = \cdot 37$	$\dfrac{5\cdot 4}{12\cdot 7} = \cdot 43$	$\dfrac{2\cdot 97}{8\cdot 02} = \cdot 37$

H and L after figures in last three columns indicate highest and lowest values.

Considering values of E for the squares 2° × 2°, 4° × 4°, 8° × 8°, viz., 15.6, 24.6, 29.2, the increase appears to be roughly as the square root of the side of the square. For the rectangular regions the differences from the square on the greater side is not very large ; so that E does *not* vary as fourth root of area. In the latter columns of the table are given values of E divided by $(x+y)^{\frac{1}{2}}$, $(x^2+y^2)^{\frac{1}{4}}$, $(x^{\frac{1}{2}}+y^{\frac{1}{2}})$ respectively; and at the foot of the respective columns the ratio of the range of values to their mean. It is seen that the variation of $E/(x^2+y^2)^{\frac{1}{4}}$ exceeds either of the others; so that the choice lies between $E/(x+y)^{\frac{1}{2}}$ and $E/(x^{\frac{1}{2}}+y^{\frac{1}{2}})$ which have equal variations. For reasons which will be discussed in the next paragraph (*see* (30.6)), it appears that the empirical form $E/(x^{\frac{1}{2}}+y^{\frac{1}{2}})$ is to be preferred. Accordingly we express the root-mean-square value of the departure of a single observed value from the mean value of a number of observations over a rectangle of sides $2x$, $2y$, as follows:—

$$\text{Root-mean-square error of representation} = E_r = 8\cdot 02\,(x^{\frac{1}{2}}+y^{\frac{1}{2}}), \qquad (29.3)$$

or, since in the region concerned the mean value of one degree is 66 miles

$$\text{Root-mean-square error of representation} = E_r = 0\cdot 7\,(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})$$
$$= 0\cdot 55\,(A^{\frac{1}{2}}+B^{\frac{1}{2}}), \qquad (29.4)$$

in which α, β are the lengths in miles of the sides of the rectangle over which the error is taken and A, B are same lengths in kilometres. It follows that

$$\text{Probable error of representation in milligals} = 0\cdot 6745\,E_r = P_r$$
$$= 0\cdot 47\,(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})$$
$$= 0\cdot 37\,(A^{\frac{1}{2}}+B^{\frac{1}{2}}). \quad \dots \quad (29.5)$$

These equations have been derived from areas of 2° square and greater and we shall have need to estimate errors of much smaller areas. According to (29.4), (29.5), the error would vanish when α and β are reduced to zero; but in fact we must find that an observed value still had an error of representation of the very point at which it was observed, namely, the error of observation, e_0. Obviously, so long as this is small compared with the errors which we have just considered, we can modify our empirical formulæ as follows. Thus

$$\text{Root-mean-square error of representation} = E_r = [e_0^2+\{0\cdot 7\,(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})\}^2]^{\frac{1}{2}} \qquad (29.6)$$

$$\text{Probable error of representation} = P_r = [\varepsilon_0^2+\{0\cdot 47\,(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})\}^2]^{\frac{1}{2}} \quad \dots \quad (29.7)$$

Thus if $e_0 = 3$ mg (or the probable error $\varepsilon_0 = 2$ mg), in a 2 degree square for which we have found a root-mean-square error of 15·6 mg, we have $(\overline{15\cdot 6}|^2+3^2)^{\frac{1}{2}} = 15\cdot 9$, which is as close to 15·6 as our empirical formula is at all likely to be precise. It is clear that it is particularly desirable to reduce e_0 in the vicinity of the origin—as can probably be done by suitable differential methods of observation—and in most cases we shall then be able to ignore e_0.

30—At first sight it was anticipated that the error of representation would vary roughly as a function of the area involved, but further consideration shows why this does not occur. It seems reasonable to regard the average gradient of Δg over a specified length as a random quantity. Consider a sphere of uniform anomalous density $\Delta\rho$ which touches the geoid internally at C and whose centre is O and radius c. At a neighbouring point P on the geoid (whose curvature is here neglected) such that angle COP is θ, if g is the gravity anomaly due to the anomalous sphere

$$\left.\begin{aligned} G &= 4\pi\rho a^3/(3a^2) = \tfrac{4}{3}\pi\rho a \\ \Delta g &= \tfrac{4}{3}\pi\,\Delta\rho c^3 \cos\theta/(c^2\sec^2\theta) = \tfrac{4}{3}\pi\,\Delta\rho c\cos^3\theta \\ \Delta g/G &= c\,\Delta\rho\cos^3\theta/(a\rho) \end{aligned}\right\} \quad . \quad . \quad . \quad (30.1)$$

If PC $= x$, then $x = c\tan\theta$ and $d\theta/dx = \cos^2\theta/c$. So

$$\frac{1}{G}\frac{d\Delta g}{dx} = -\frac{3c}{a}\frac{\Delta\rho}{\rho}\cos^2\theta\sin\theta\,\frac{1}{c}\cos^2\theta = -\frac{3}{a}\frac{\Delta\rho}{\rho}\cos^4\theta\sin\theta. \quad . \quad (30.2)$$

Now $d\Delta g/dx$ is the gradient of Δg in direction x, and it is independent of c but varies as $\Delta\rho$. If then we consider the density anomalies of the crust as made up of a set of spheres of uniform anomaly all touching the geoid, the gradient of Δg depends on the amount of the density anomaly. It is reasonable to regard the departures from normal density to have a random distribution, in which case the horizontal gradients of gravity will also have a random distribution.

Consider then a number of points 0, 1, 2, . . . n along a line at equal unit intervals and suppose the gradient between $(r-1)$ and r is u_r; and let U be the probable value of u_r. The anomaly of gravity at point r will be

$$\Delta g_r = \Delta g_0 + u_1 + u_2 \ldots + u_r,$$

and the mean value of this for all the points 0 to r is Γ, where

$$(n+1)\,\Gamma = (n+1)\,g_0 + n\,.\,u_1 + (n-1)\,u_2 + \ldots + (n-r+1)\,u_r \ldots + u_n. \quad (30.3)$$

Hence

$$\Delta g_r - \Gamma = \frac{1}{n+1}\{u_1 + 2u_2 + \ldots + ru_r - (n-r)\,u_{r+1} \ldots - u_n\}, \qquad (30.4)$$

of which the probable value is

$$\frac{U}{n+1}\left\{\sum_1^r r^2 + \sum_1^{n-r} r^2\right\}^{\frac{1}{2}}.$$

The probable error of representation is the root-mean-square sum of this for all values of r from 1 to n; so

$$\begin{aligned} P_r &= \frac{U}{n+1}\left\{\frac{2}{n}\,(n\,.\,1^2 + \overline{n-1}\,|\,2^2 + \ldots)\right\}^{\frac{1}{2}} = \frac{U}{n+1}\left\{2\,(n+1)\sum_1^n r^2 - 2\sum_1^n r^3\right\}^{\frac{1}{2}} \\ &= \frac{U}{n+1}\left\{\frac{2}{n}\cdot\frac{n\,(n+1)^2\,(2n+1)}{6} - \frac{2}{n}\cdot\frac{n^2\,(n+1)^2}{4}\right\}^{\frac{1}{2}} = U\left\{\frac{2n+1}{3} - \frac{n}{2}\right\}^{\frac{1}{2}} \\ &= U\left\{\frac{n}{6} - \frac{1}{2}\right\}^{\frac{1}{2}} \doteqdot U\sqrt{\frac{n}{6}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30.5) \end{aligned}$$

If we now consider a set of n by l squares of unit sides, it is at once apparent that the gradients in the two orthogonal directions are conditioned; for the sum of the four gradients round each elementary square must total zero. There are now $(n+1) \times (l+1)$ points, corners of the squares, related by a like number of gradients among which there are nl conditions. We may get a rough idea of the error by considering the case of $(n+1)(l+1) - nl = n + l + 1$ independent gradients. If we take the two lines at right angles emanating from an outside corner of the square system, these may be considered to have $n + l$ independent gradients; and so should give an error of representation of approximately the same amount as the value for the network of $(n+1)$ $(l+1)$ gradients connected by nl conditions. Now, as these lines of length l and n are considered independent except at their junction point, they will give rise to a representative error of the same amount as a single line of their combined length; so from (30.5) we have $P_r = U\{(n+l)/6\}^{\frac{1}{2}}$. It seems clear that the estimate derived from two simple lines at right angles will err on the side of being too small to represent a rectangle of sides l and n. Moreover, for a long narrow strip, in which n/l is large, the effect of l would be trivial, for two adjacent parallel lines will give almost identical results. It is also to be remembered that the expression for P_r must be such as to reduce to $U\,(n/6)^{\frac{1}{2}}$ when $l = 0$ and to $U\,(l/6)^{\frac{1}{2}}$ when $n = 0$. The form $U\,(\sqrt{n/6} + \sqrt{l/6})$ meets these several desiderata.

Hence we may expect that the representative error for a rectangle of side lengths α, β miles will be fairly given by the empirical formula

$$P_r = K\,(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}), \qquad (30.6)$$

in which K is constant, at least for a large region. We have already found the numerical value $0{\cdot}47$ for K resulting from data obtained in India, *vide* (29.5). A more typical value for the whole earth may be found later. It may be noticed, however, that for a square of the same area as that of the whole earth in which $\alpha = \beta = (\text{earth's area})^{\frac{1}{2}} = 10^4\,\sqrt{2}$, we get from (29.5)

$$P_r = 0{\cdot}47 \times 2 \times 10^2 \times 2^{\frac{1}{4}} = 113\text{mg}. \qquad (30.7)$$

So far as observations of gravity have gone, this is quite a reasonably probable value of anomaly of gravity from the standard formula value: which is the error of representation for the whole earth of a single determination of gravity. This supports the idea on which the present paragraph is based, viz., that the horizontal gradients of gravity anomaly may be regarded as of random distribution.

31—Much depends on whether errors are of a random character or appreciably systematic. What we have derived as error of representation is the probable value of a random error; but if stations were injudiciously sited the error might be in part systematic. It would certainly be unwise to site stations predominantly on coast-lines, or on small islands in deep oceans, where systematic anomalies of gravity are already expected to occur; and with further observational results available it may become possible to lay down more complete recommendations for the siting of stations. Ordinarily we should wish to have a gravity station at the centre of each region of quadrature; and as the regions of the general distribution are to be equal this would lead to a symmetrical distribution which would generally be random in relation to the topography. In considering the number of separate regions which are desirable, we shall assume that systematic error is avoided, so that E_r can be regarded as a random error.

32—The average probable error of observation is of the order 2 mg for land observations and 4 mg for sea observations,* and the latter figure shows a tendency to decrease in recent work. So 3 mg may be taken as a general estimate of probable error of observation, ε_0, for both land and sea stations, the root-mean-square error, e_0, being 4 mg. This error is random.

The main source of error of reduction, e_1, will be due to uncertainty as to height of station—which applies only to land stations. An error of 100 feet in height will cause an error of 10 mg. This is rather an extreme height error for any visited station, at which some form of hypsometric measurements would assuredly be made. The error in computing the effect of compensated topography is considerably less for the same height error. In unmapped regions perhaps as much as 5 mg might arise from this cause. As regards the ocean areas, it is not essential to reduce observations on the basis of compensation, seeing that there is no matter protruding here outside the geoid; and the object of reducing in this manner is simply to reduce the error of representation. What is important here as elsewhere is to make sure that any assumptions made regarding ocean depths are used systematically on all occasions. The same applies to heights of land regions. Finally, when we revert from the compensated geoid to the real geoid, these reductions will be restored, and any error will be cancelled. In view of this, I think that the error of reduction can be expected to average no more than 5 mg, and to be of a random character.

The error of apparatus, e_a, is likely to present difficulties so far as this error may be largely systematic. One might have expected that with proper standardization such error would be avoided. However, distressingly large discrepancies have in the past occurred when various apparatuses have been compared, and satisfactory explanation has not been forthcoming. Pendulums change their period for rather cryptic reasons. This difficulty will not be discussed further here, where it is sufficient to say that especial care is needed to guard against it in any world gravity survey, which might well be marred by errors of this nature.

In regions where the gravity stations are widely separated, the dominant error is that of representation; while for local surveys it is reasonable to suppose that instrumental error is locally constant, the adjacent stations having in general been occupied about the same time and with the same apparatus. In the immediate neighbourhood of an origin, at which the deviation of geoidal vertical is sought, it is to be expected that special measures (*vide* § 43) will be taken to make the differential error of observation a minimum. Hence in all cases we shall be concerned mainly with the error of representation exhibited in § 29.

* Vening Meinesz, "Gravity Expeditions at Sea, 1923–30," p. 101.

Partition of the Earth for World Gravity Stations

33—Suppose the earth's surface divided into n regions, approximately equal, for which

$$\Delta\omega = \int d\omega \text{ over the region} \doteqdot \frac{4\pi}{n} \quad \ldots \quad (33.1)$$

$$= \iint \sin\psi \, d\psi \, d\chi = -\Delta\chi \, \Delta\cos\psi = 2\Delta\chi \sin\psi \sin\frac{\Delta\psi}{2} \doteqdot \Delta\chi \, \Delta\psi \sin\psi, \quad (33.2)$$

the limits being $\chi \pm \frac{1}{2}\Delta\chi$, $\psi \pm \frac{1}{2}\Delta\psi$. The quantity $\Delta\chi$ is clearly $2\pi/K$, where K is some integer, being the number of sub-divisions of the zone ψ_k. As $\Delta\omega$ is to be sensibly the same for all regions it follows that $\sin\psi_k/K$ must be sensibly constant; it cannot be absolutely constant as K must be an integer. Hence

$$K = \text{integer nearest to } A \sin\psi_k. \quad \ldots \quad (33.3)$$

If there are N zones, keeping $\Delta\psi$ always the same

$$\Delta\psi = \pi/N, \quad \ldots \quad (33.4)$$

and

$$\Delta\omega \doteqdot \frac{4\pi}{n} \doteqdot \frac{2\pi}{K} \cdot \frac{\pi}{N} \sin\psi_k = \frac{2\pi^2}{NA}, \quad \ldots \quad (33.5)$$

so that

$$A = \frac{n\pi}{2N} \qquad K = \frac{\pi n}{2N} \sin\psi_k. \quad \ldots \quad (33.6)$$

As we have taken all the $\Delta\psi$'s equal, the mean value of ψ for the k^{th} zone is

$$\psi_k = \frac{2k-1}{2} \Delta\psi = \frac{2k-1}{2} \frac{\pi}{N}, \quad \ldots \quad (33.7)$$

where k ranges from 1 to N. Near the equator the regions will be practically square if $K_{\frac{\pi}{2}} = 2N$. Hence from (33.6) (33.7)

$$N = \left(\frac{\pi n}{4}\right)^{\frac{1}{2}} \qquad \psi_k = (2k-1)\left(\frac{\pi}{n}\right)^{\frac{1}{2}}, \quad \ldots \quad (33.8)$$

and

$$K = (n\pi)^{\frac{1}{2}} \sin\psi_k = 2N \sin\psi_k = (2k-1)\,\pi \,.\, \frac{\sin\psi_k}{\psi_k}, \quad \ldots \quad (33.9)$$

so that near the pole, $K \doteqdot (2k-1)\pi$.

As a numerical example take $\Delta\psi = 5°$, so that $N = 36$ and

$$K = 2N \sin\psi_k = 72 \sin\psi_k.$$

We get the following values :—

k	1	2	3	4	5	6	7	8	9
ψ_k	2°·5	7°·5	12°·5	17°·5	22°·5	27°·5	32°·5	37°·5	42°·5
K	3	9	16	22	28	33	39	44	49
k	10	11	12	13	14	15	16	17	18
ψ_k	47°·5	52°·5	57°·5	62°·5	67°·5	72°·5	77°·5	82°·5	87°·5
K	53	57	61	64	67	69	70	71	72

$$\sum_0^{2\pi} \mathrm{K} = 2 \times 827 = 1654 = n. \quad \text{. . . . .} \quad (33.10)$$

Taking the International values of earth's semi-major axis, $a' = 6378{\cdot}388$ km $= 3963{\cdot}351$ miles and $\varepsilon = 1/297$, the corresponding mean radius $a = 6371{\cdot}229$ km $= 3958{\cdot}901$ miles, earth's area is $5{\cdot}10101 \times 10^8$ km^2 $= 1{\cdot}969514 \times 10^8$ miles^2. When divided into 1654 equal regions, each $= (555{\cdot}34 \text{ km})^2 = (345{\cdot}07 \text{ miles})^2$. So for the application of (29.5) we have

$$\mathrm{A}^{\frac{1}{2}} = \mathrm{B}^{\frac{1}{2}} = 23{\cdot}57 \qquad \alpha^{\frac{1}{2}} = \beta^{\frac{1}{2}} = 18{\cdot}58. \quad \text{. .} \quad (33.11)$$

Geoidal Anomalies at an Origin of Geodetic Survey and Probable Errors of Deduced Values

34—Consider now the evaluation of Δr, ξ, η at an origin of survey, P′. From (20·2) and (22·3) we have

$$\left.\begin{aligned} \Delta r &= \frac{a}{4\pi \mathrm{G}} \int \Delta g_s f \, d\omega \\ \begin{matrix}\eta\\ \xi\end{matrix} &= -\frac{\operatorname{cosec} 1''}{4\pi \mathrm{G}} \int \Delta g_s \frac{df}{d\psi} \begin{matrix}\cos\\ \sin\end{matrix} \chi \, d\omega \end{aligned}\right\} \quad \text{. .} \quad (34.1)$$

We shall suppose that values of Δg are known at one point in each of the n regions of § 33, and shall consider the probable errors of Δr, η, ξ. We shall assume that each observed value of Δg applies to the whole of the region concerned with a probable error of representation as given by (29.5), which by means of (33.11) is seen to be $0{\cdot}47 \times 2 \times 18{\cdot}6 = 17{\cdot}5$ mg. For simplicity we assume that P′ is at the pole of the system of division of § 33, which assumption will not seriously affect the derivation of the probable error. Put

$$\left.\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \Delta g_s \, d\chi &= \Delta g_k \\ \frac{1}{2\pi} \int_0^{2\pi} \Delta g_s \begin{matrix}\cos\\ \sin\end{matrix} \chi \, d\chi = \left(\Delta g_s \begin{matrix}\cos\\ \sin\end{matrix} \chi \right)_k &= \Delta h_k \end{aligned}\right\}, \quad \text{.} \quad (34.2)$$

the suffix k indicating that the result applies to the kth zone. Then

$$\left.\begin{aligned} \Delta r &= \frac{a}{2G}\int_0^{\pi} \Delta g_k f \sin\psi\, d\psi \\ {\eta \atop \xi} &= \frac{\operatorname{cosec} 1''}{2G}\int_0^{\pi} \Delta h_k \frac{df}{d\psi}\sin\psi\, d\psi \end{aligned}\right\} \quad . \quad . \quad (34.3)$$

In the case under consideration Δg_k and Δh_k are to be considered constant over a zone, *vide* (33.7), bounded by $\psi = (k-1)\,\Delta\psi$ and $\psi = k\Delta\psi$, where $\Delta\psi = \pi/N$. Put

$$\frac{1}{\Delta\psi}\int_{(k-1)\Delta\psi}^{k\Delta\psi} f\sin\psi\, d\psi = F_k, \qquad \frac{1}{\Delta\psi}\int_{(k-1)\Delta\psi}^{k\Delta\psi} \frac{df}{d\psi}\sin\psi\, d\psi = H_k. \quad . \quad . \quad (34.4)$$

Then

$$\Delta r = \frac{a}{2G}\frac{\pi}{N}\sum_1^N \Delta g_k F_k, \qquad {\eta \atop \xi} = \frac{\operatorname{cosec} 1''}{2G}\frac{\pi}{N}\sum_1^N \Delta h_k H_k. \quad . \quad . \quad (34.5)$$

35—It will be found that for a satisfactory solution the number of gravity determinations in the neighbourhood of P′ must be greater than that laid down in § 33. We shall consider the case of the vicinity later (§ 40). For the general world survey we have from (34.5)

$$\text{Probable error of } \Delta r = \frac{aP_r}{2G}\frac{\pi}{N}\left(\sum_1^N \frac{F_k^2}{K}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35.1)$$

$$\text{Probable error of } {\eta \atop \xi} = \frac{\operatorname{cosec} 1''\,.\,P_r}{2\sqrt{2}G}\cdot\frac{\pi}{N}\left(\sum_1^N \frac{H_k^2}{K}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (35.2)^*$$

A factor $1/\sqrt{2}$ is introduced in (35.2) as the p.e. of Δh_k (*vide* (34.2)) is $P_r/\sqrt{2}$). It is to be remarked that since K varies as N, *vide* (33.9), the terms within the brackets, excluding the portion relating to the immediate vicinity of P′ are not seriously affected by changes in N. Introducing numerical values we may conveniently write the last two equations

$$\text{Probable error of } \Delta r \text{ in feet} = 16\cdot3\left(\frac{P_r}{17\cdot5}\right)\left(\frac{36}{N}\right)\left(\sum_1^N \frac{F_k^2}{K}\right)^{\frac{1}{2}}, \quad . \quad . \quad (35.3)$$

$$\text{Probable error of } {\eta \atop \xi} = 0''\cdot114\left(\frac{P_r}{17\cdot5}\right)\left(\frac{36}{N}\right)\left(\sum_1^N \frac{H_k^2}{K}\right)^{\frac{1}{2}}. \quad (35.4)$$

36—For the computation of numerical values we write (20.4)

$$f = \operatorname{cosec}\tfrac{1}{2}\psi + 1 - 6\sin\tfrac{1}{2}\psi - B\cos\psi,$$

where

$$B = 5 + 3\log\{\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36.1)$$

* In § 44 we shall see that P_r must be replaced by $\frac{1}{2}P_r$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Then

$$\frac{dB}{d\psi}=\frac{3}{2}\cos\tfrac{1}{2}\psi\left(\frac{1}{\sin\frac{1}{2}\psi}+\frac{1-\sin\frac{1}{2}\psi}{\cos^2\frac{1}{2}\psi}\right)=\frac{3}{\sin\psi}(\cos\psi+\sin\tfrac{1}{2}\psi), \quad . \quad . \quad (36.2)$$

$$f\sin\psi=-\cos\tfrac{1}{2}\psi+\sin\psi+3\cos 3\psi/2-\tfrac{1}{2}B\sin 2\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36.3)$$

$$\frac{df}{d\psi}=-\tfrac{1}{2}\cos\tfrac{1}{2}\psi\,\mathrm{cosec}^2\,\tfrac{1}{2}\psi-3\sin 3\psi/2\,\mathrm{cosec}\,\psi$$
$$-3\cot\psi\cos\psi+B\sin\psi, \quad (36.4)$$

$$\frac{df}{d\psi}\sin\psi=-\cos^2\tfrac{1}{2}\psi\,\mathrm{cosec}\,\tfrac{1}{2}\psi-3\sin 3\psi/2-\tfrac{3}{2}-\tfrac{3}{2}\cos 2\psi\;B\sin^2\psi, \quad . \quad . \quad (36.5)$$

$$\frac{df}{d\psi}\sin^2\psi=-3\cos\tfrac{1}{2}\psi-\tfrac{1}{2}\cos 3\psi/2+\tfrac{3}{2}\cos 5\psi/2-\tfrac{3}{4}(\sin\psi+\sin 3\psi)+B\sin^3\psi. \quad (36.6)$$

For larger values of ψ it is convenient to replace ψ by $\pi-2\theta$ when

$$\frac{df}{d\psi}=-\tfrac{1}{2}\tan\theta\sec\theta-3\sin\theta+B\sin 2\theta+6\cot 2\theta\sin\tfrac{1}{2}\theta\sin 3\theta/2$$

$$f\sin\psi=-\tfrac{1}{2}B\sin 4\theta-\sin\theta+\sin 2\theta-3\sin 3\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36.7)$$

For computation

$$B=5+6\cdot9077\log_{10}\{\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\}$$
$$=6\cdot9077\,[0\cdot72383+\log_{10}\{\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\}]$$
$$\log_{10}B=0\cdot83934\log_{10}[0\cdot72383\log_{10}\{\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\}]. \quad . \quad (36.8)$$

With the help of the above, values of $f\sin\psi$ and $df/d\psi\sin\psi$ have been computed. The latter quantity is infinite when $\psi=0$. It may be represented approximately (within 2% up to $\psi=15°$) for small values of ψ by

$$\frac{df}{d\psi}\sin\psi=\frac{114}{\psi°}+3\cdot8. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36.9)$$

37—After tabular values of $f\sin\psi$ and $df/d\psi\sin\psi$ have been found for all values of $k\Delta\psi=5k°$ from 0° to 180°, we proceed as follows for F_k and H_k. Denote $(f\sin\psi)_k$ or $(df/d\psi\sin\psi)_k$ by u_0 and for successive decreasing values of suffix $(k-1)$, $(k-2)$ . . . by u_1, u_2 . . . . If the successive order differences are denoted by Δ_1, Δ_2 . . . then

$$u_x=u_0+x\,\Delta_1+\frac{x\,(x-1)}{2}\,\Delta_2+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.1)$$

$$\frac{1}{\Delta\psi}\int_{(k-1)\,\Delta\psi}^{k\Delta\psi}u\,d\psi=\int_0^1 u_x\,dx=u_0+\frac{1}{2}\,\Delta_1-\frac{1}{12}\,\Delta_2+\frac{1}{24}\,\Delta_3-\frac{19}{720}\,\Delta_4\ldots, \quad (37.2)$$

by means of which F_k, H_k can be computed, *vide* (34.4).

For the case when k, ψ are small, the successive differences are large and finally divergent. Using (36.9)

$$\mathrm{H}_k = \frac{1}{\Delta\psi}\int_{(k-1)\Delta\psi}^{k\Delta\psi} \frac{df}{d\psi}\sin\psi d\psi = \frac{1}{\Delta\psi}[114\log_e\psi + 3\cdot 8\,\psi] = \frac{262}{\Delta\psi^\circ}\log_{10}\frac{k}{k-1} + 3\cdot 8. \tag{37.3}$$

38—In Table IV are given values of $f \sin \psi$, $df/d\psi \sin \psi$ each with their first- and second-order differences, F_k, H_k, $\mathrm{F}_k^2/\mathrm{K}$, $\mathrm{H}_k^2/\mathrm{K}$, all for values of $\psi = 5k^\circ$, where k has all integral values from 0 to 36. Values of K are taken from § 33. The progressive totals $\Sigma\,(\mathrm{F}_k^2/\mathrm{K})$, $\Sigma\,(\mathrm{H}_k^2/\mathrm{K})$ are given in some cases, the summation being taken from the antipodes. We see that the value of $\mathrm{F}_k^2/\mathrm{K}$, which applies to a zone $5k^\circ - 5\,(k-1)^\circ$, increases rapidly as k approaches unity, but it does not become infinite. The sum $\sum\limits_1^{36}(\mathrm{F}_k^2/\mathrm{K})$ is $3\cdot 879$; so from (33.11) and (35.3) it follows that the probable error of Δr is $0\cdot 47 \times 18\cdot 58 \times 2 \times (3\cdot 879)^{\frac{1}{2}} = 34$ feet. This is approximately $1\cdot 5 \times 10^{-6}\,.\,a$, where a is the mean radius of the earth. The precision can be increased by additional local stations. It is, however, to be remembered that the error in a is of this same order.

39—The probable errors of η, ξ present a considerably different case. Here the values of $\mathrm{H}_k^2/\mathrm{K}$ increase very rapidly for small diminishing values of k and become infinite when $k = 1$. We must consider the nature of Δh_k—*see* (34.2)—in the neighbourhood of P′, where $\psi = 0$. Suppose that we express Δg_s as a series of harmonic terms, so that

$$\Delta g_s = \Sigma\, v_n, \tag{39.1}$$

where

$$v_n = \mathrm{A}_n\mathrm{P}_n + \sin\psi\frac{d\mathrm{P}_n}{dp}(\mathrm{B}_n\cos\chi + b_n\sin\chi)$$
$$+ \sin^2\psi\frac{d^2\mathrm{P}_n}{dp^2}(\mathrm{C}_n\cos 2\chi + c_n\sin 2\chi) + \dots \tag{39.2}$$

Then

$$\int_0^{2\pi} v_n\,d\chi = 2\mathrm{A}_n\mathrm{P}_n \qquad \int_0^{2\pi} v_n\cos\chi\,d\chi = \mathrm{B}_n\sin\psi\frac{d\mathrm{P}_n}{dp}. \tag{39.3}$$

Hence

$$\Delta g_k = \Sigma\,\mathrm{A}_n\mathrm{P}_n \qquad \Delta h_k = \tfrac{1}{2}\sin\psi\,\Sigma\,\mathrm{B}_n\frac{d\mathrm{P}_n}{dp}. \tag{39.4}$$

It follows that Δh_k is essentially zero when $k = 0$ and so it is wrong to burden it with an observation error at that point. It is clear from (39.2) that we may write

$$\Delta g_k = \mathrm{L}_0 + \mathrm{L}_2\sin^2\psi + \mathrm{L}_4\sin^4\psi + \dots \tag{39.5}$$

$$\Delta h_k = \mathrm{M}_1\sin\psi + \mathrm{M}_3\sin^3\psi + \dots \tag{39.5}$$

and one method of proceeding would be to determine the M coefficients from the observational data—a process which may be followed for the immediate vicinity. For the middle distances we may employ (22.10). Thus between radii R_I and R_0

$$\delta\xi'' = -\frac{\operatorname{cosec} 1''}{4\pi G}\left\{\int_{R_I}^{R_0}\int_0^{2\pi}\frac{\partial\,\Delta g}{\partial x}\,.\,\phi\,d\rho\,d\chi - \int_0^{2\pi}[\Delta g\phi]_{R_I}^{R_0}\sin\chi\,d\chi\right\}$$

$$= -\frac{\operatorname{cosec} 1''}{4\pi G}(I_1+I_2) = -0''{\cdot}0167\,(I_1+I_2), \quad . \;. \;. \quad (39.6)$$

where $\phi = f\sin\psi$, $\rho = a\sin\psi$, $x = \rho\sin\chi$, $y = \rho\cos\chi$ and ξ is the deviation of the geoidal normal from the spheroidal normal in the direction in which x is measured.

40—Considering a narrow ring bounded by radii P, P′, the corresponding portion of the surface integral I_1 is

$$\Delta I_1 = \left(\frac{\phi}{\rho}\right)_a\int_\rho^{\rho'}\int_0^{2\pi}\frac{du}{\partial x}\,dS = \left(\frac{\phi}{\rho}\right)_a\left[\int_{\rho'} u\,dy - \int_\rho u\,dy\right], \quad . \;. \;. \quad (40.1)$$

in which u is written for Δg, dS is an element of area and so equal to $dxdy$, $(\phi/\rho)_a$ lies between ϕ'/ρ' and ϕ/ρ, and $\int_\rho$ indicates integration round the circle ρ. Now along this circle $dy = \rho\sin\chi\,d\chi$ so

$$\Delta I_1 = \left(\frac{\phi}{\rho}\right)_a\left[\rho'\int_{\rho'} u'\sin\chi\,d\chi - \rho\int_\rho u\sin\chi\,d\chi\right], \quad . \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \quad (40.2)$$

$$I_1 = \Sigma\,\Delta I_1 = \Sigma\left(\frac{\phi}{\rho}\right)_{m-\frac{1}{2}}\left[\rho_m\int_{\rho_m} u_m\sin\chi\,d\chi - \rho_{m-1}\int_{\rho_{m-1}} u_{m-1}\sin\chi\,d\chi\right]$$

$$= \left(\frac{\phi}{\rho}\right)_{N+\frac{1}{2}}\rho_N\int u_N\sin\chi\,d\chi - \left(\frac{\phi}{\rho}\right)_{\frac{1}{2}}\rho_0\int u_0\sin\chi\,d\chi - \sum_1^{n-1}\left[\left(\frac{\phi}{\rho}\right)_{m+\frac{1}{2}} - \left(\frac{\phi}{\rho}\right)_{m-\frac{1}{2}}\right]$$

$$\times\,\rho_m\int u_m\sin\chi\,d\chi\,. \quad (40.3)$$

In this the suffix m indicates a value on radius ρ_m, but suffix $m-\frac{1}{2}$ indicates a value, not precisely defined, intermediate to the values on radii ρ_m and ρ_{m-1}. It is to be noted, however, that if a value on too large a radius is taken, the effect is partly cancelled by the adjacent element. We may accordingly, without serious error, give any reasonable convenient interpretation to suffix $m-\frac{1}{2}$. In the region with which we are concerned, $\phi \not> 20°$, the value of ϕ ranges between 2.43 at 7° and 1·88 at 20°, and so never varies by more than 13% from the medial value 2.15. In discussing the probable errors and best distribution of gravity stations in this region we shall treat ϕ as constant and equal to 2·15; and we shall put

$$\rho_{m-\frac{1}{2}} = F\,(m-\tfrac{1}{2}), \quad . \;. \;. \;. \;. \;. \;. \;. \;. \;. \quad (40.4)$$

where $F\,(m) = \rho_m$.

We shall also consider that circle ρ_m is divided into K_m sections of length s_m, where

$$K_m s_m = 2\pi\rho_m. \quad . \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \quad (40.5)$$

3 I 2

TABLE IV—PART I. N = 36

ψ°	φ=f sin ψ	Δ_1	Δ_2	F_k	F_k^2/K	$\Sigma F_k^2/K$	k	K	df/dψ sin ψ	Δ_1	Δ_2	H_k	H_k^2/K	$\Sigma H_k^2/K$
180°	+0·000	+0·266		+0·134	0·0060	—	36	3	+0·000	+0·049		+0·02	+0·000	—
175	0·266	0·252	−0·014	0·394	0·0173	—	35	9	0·049	+0·145	+0·096	+0·11	0·001	—
170	0·518	0·222	−0·030	0·533	0·0178	—	34	16	0·194	+0·233	+0·088	+0·30	0·006	—
165	0·740	0·182	−0·040	0·836	0·0319	—	33	22	0·427	+0·312	+0·079	+0·58	0·015	—
160	0·922	0·128	−0·054	0·991	0·0351	—	32	28	0·739	+0·374	+0·072	+0·92	0·030	—
155	1·050	0·068	−0·060	1·084	0·0356	—	31	33	1·113	+0·420	+0·046	+1·32	0·053	—
150	1·118	+0·001	−0·067	1·124	0·0325	—	30	39	1·533	+0·445	+0·025	+1·76	0·080	—
145	1·119	−0·067	−0·068	1·091	0·0271	—	29	44	1·978	+0·448	+0·003	+2·20	0·110	—
140	1·052	−0·136	−0·069	0·989	0·0200	—	28	49	2·426	+0·425	−0·023	+2·64	0·143	—
135	0·916	−0·201	−0·065	0·820	0·0113	—	27	53	2·851	+0·385	−0·040	+3·05	0·175	—
130	0·715	−0·257	−0·056	0·590	0·0061	—	26	57	3·236	+0·320	−0·065	+3·40	0·203	—
125	0·458	−0·303	−0·046	+0·309	0·0016	—	25	61	3·556	+0·236	−0·084	+3·68	0·222	—
120	+0·155	−0·338	−0·035	−0·012	0·0000	0·2423	24	64	3·792	+0·133	−0·103	+3·87	0·235	—
115	−0·183	−0·357	−0·019	−0·361	0·0019	—	23	67	3·925	+0·018	−0·115	+3·94	0·232	—
110	−0·540	−0·360	−0·003	−0·721	0·0076	—	22	69	3·943	−0·106	−0·124	+3·90	0·221	—
105	−0·900	−0·346	+0·014	−1·076	0·0165	—	21	70	3·837	−0·238	−0·132	+3·73	0·199	—
100	−1·246	−0·315	0·031	−1·408	0·0280	—	20	71	3·599	−0·368	−0·130	+3·43	0·166	—
95	−1·561	−0·267	0·048	−1·547	0·0332	—	19	72	3·231	−0·495	−0·127	+2·99	0·125	—
90	−1·828	−0·205	0·062	−1·937	0·0522	—	18	72	2·736	−0·609	−0·114	+2·44	0·083	—
85	−2·033	−0·131	0·074	−2·106	0·0629	—	17	71	2·127	−0·720	−0·111	+1·77	0·044	—
80	−2·164	−0·046	0·085	−2·195	0·0689	—	16	70	1·407	−0·806	−0·086	+1·01	0·015	—
75	−2·210	+0·046	0·092	−2·195	0·0695	—	15	69	+0·601	−0·876	−0·070	+0·17	0·000	2·358
70	−2·164	0·140	0·094	−2·102	0·0660	—	14	67	−0·275	−0·924	−0·048	−0·74	0·008	2·366
65	−2·024	0·233	0·093	−1·915	0·0573	—	13	64	−1·199	−0·949	−0·025	−1·67	0·044	2·410
60	−1·791	0·319	0·086	−1·638	0·0440	—	12	61	−2·148	−0·952	−0·003	−2·63	0·114	2·524
55	−1·472	0·398	0·079	−1·278	0·0286	—	11	57	−3·100	−0·942	+0·010	−3·57	0·225	2·749
50	−1·074	0·460	0·062	−0·849	0·0138	—	10	53	−4·042	−0·915	+0·027	−4·45	0·375	3·124
45	−0·614	0·505	0·045	−0·363	0·0027	—	9	49	−4·957	−0·882	+0·033	−5·40	0·597	3·721
40	−0·109	0·529	+0·024	+0·156	0·0006	0·7960	8	44	−5·839	−0·862	+0·020	−6·27	0·897	4·618
35	+0·420	0·528	−0·001	+0·686	0·0121	0·8081	7	39	−6·701	−0·867	−0·005	−7·13	1·30	5·92
30	0·948	0·497	−0·031	+1·201	0·0438	0·8519	6	33	−7·568	−0·951	−0·084	−8·03	1·96	7·88
25	1·445	0·437	−0·060	+1·669	0·0995	0·9514	5	28	−8·519	−1·186	−0·235	−9·18	3·02	10·90
20	1·882	0·342	−0·095	+2·060	0·193	1·144	4	22	−9·705	−1·825	−0·639	−10·5	5·01	15·91
15	2·224	0·205	−0·137	+2·300	0·331	1·475	3	16	−11·53	−3·61	−1·78	−13·1	10·7	26·6
10	2·429	0·004	−0·201	+2·452	0·668	2·143	2	9	−15·14	−11·14	−7·53	−19·5	42·3	68·9
5	2·433	−0·433	−0·437	+2·282	1·736	3·879	1	3	−26·28	−∞	−∞	−∞	∞	∞
0	+2·000								−∞					

Part II. N = 180

15°	2·224	+0·054		—11·53	—0·51	
14	2·278	+0·047	—0·007	—12·04	—0·59	—0·08
13	2·325	+0·042	—0·005	—12·63	—0·70	—0·11
12	2·367	+0·035	—0·007	—13·33	—0·82	—0·12
11	2·402	+0·027	—0·008	—14·15	—0·99	—0·17
10	2·429	+0·020	—0·007	—15·14	—1·20	—0·21
9	2·449	+0·011	—0·009	—16·34	—1·52	—0·32
8	2·460	+0·003	—0·008	—17·86	—2·01	—0·49
7	2·463	—0·009	—0·012	—19·87	—2·67	—0·66
6	2·454	—0·021	—0·012	—22·54	—3·74	—1·07
5	2·433	—0·038	—0·017	—26·28	—5·66	—1·82
4	2·395	—0·056	—0·018	—31·94	—9·49	—3·83
3	2·339	—0·072	—0·016	—41·43	—19·03	—9·54
2	2·267	—0·090	—0·018	—60·46	—55·2	—36·2
1	2·177	—0·177	—0·087	—117·7	—∞	—∞
0	2·000			—∞		

Then

$$J = I_1 - \left(\frac{\phi}{\rho}\right)_{n-\frac{1}{2}} \rho_n \int u_n \sin\chi \, d\chi + \left(\frac{\phi}{\rho}\right)_{\frac{1}{2}} \rho_0 \int u_0 \sin\chi \, d\chi$$

$$\doteq 2\cdot 15 \sum_1^{n-1} \left(\frac{1}{\rho_{m-\frac{1}{2}}} - \frac{1}{\rho_{m+\frac{1}{2}}}\right) \rho_m \,.\, 2\pi \text{ (average value of } u_m \sin\chi \text{ over radius } \rho_m) \,. \quad (40.6)$$

We shall further suppose that the length s_m is chosen so that

$$s_m = \rho_{m+\frac{1}{2}} - \rho_{m-\frac{1}{2}}, \quad \ldots\ldots\ldots \quad (40.7)$$

which leads to the division of the area into quasi-squares of side s_m. Then

$$J = 2\cdot 15 \sum_1^{n-1} \frac{2\pi\rho_m s_m}{\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} \text{ (average value of } u_m \sin\chi \text{ over } \rho_m) \,. \quad (40.8)$$

The average $u_m \sin\chi$ is the mean of K_m terms, each the average of $u_m \sin\chi$ over an arc of length s_m. The error of representation of one gravity station on this arc is the probable difference of the value at this station from the mean and is, from (29.5), (29.6), $\{\varepsilon_0^2 + \overline{0\cdot 47}|^2 s_m\}^{\frac{1}{2}}$, where ε_0 is the probable error of observation. Remembering that the average value of $\sin^2\chi$ is $\frac{1}{2}$, we see that the

$$\text{Probable error of } \textit{one} \text{ element of } J = \frac{2\cdot 15}{\sqrt{2}} \cdot \frac{2\pi\rho_m s_m}{K_m \rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} \{\varepsilon_0^2 + \overline{0\cdot 47}|^2 s_m\}^{\frac{1}{2}}$$

$$= 1\cdot 52 \frac{s_m^2}{\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} \{\varepsilon_0^2 + \overline{0\cdot 47}|^2 s_m\}^{\frac{1}{2}}, \quad \ldots \quad (40.9)$$

using (40.5).

Reverting to (39.6), (40.6)

$$I_1 + I_2 = J. \quad \ldots\ldots\ldots\ldots\ldots \quad (40.10)$$

$$\delta\xi = -0''\cdot 0167\, J. \quad \ldots\ldots\ldots \quad (40.11)$$

The bounding outer and inner radii are $\rho_{N+\frac{1}{2}} = R_0$ and $\rho_{\frac{1}{2}} = R_I$. Introducing the the numerical factor $0\cdot 0167$ in (40.9) we see that

$$\text{Probable error of } \delta\xi'' \text{ due to one element of quadrature } s_m^2 = 0''\cdot 0254 \frac{s_m^2}{\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} \{\varepsilon_0^2 + \overline{0\cdot 47}|^2 s_m\}^{\frac{1}{2}}. \quad (40.12)$$

Optimum Distribution of Local Gravity Stations

41—For the optimum distribution of gravity stations it is clear that the contribution of each station to the probable error must be the same. Hence for the ideal distribution

$$\frac{s_m^2}{\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} \{\varepsilon_0^2 + \overline{0\cdot 47}|^2 s_m\}^{\frac{1}{2}} = \text{constant for all values of } m. \quad (41.1)$$

Case I—If ε_0 is large compared with $0\cdot 47\ s_m^{\frac{1}{2}}$ this requires that

$$s_m^2/(\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}) = \text{constant}, \qquad (41.2)$$

and using (40.7) it follows that $\rho_{m+\frac{1}{2}}/\rho_{m-\frac{1}{2}}$ is constant, so that

$$\rho_{m+\frac{1}{2}} = \mathrm{R_I}\,.\,\gamma^m \qquad \gamma = \left(\frac{\mathrm{R_0}}{\mathrm{R_I}}\right)^{1/\mathrm{N}}, \qquad (41.3)$$

where γ is a numerical constant. Then from (40.7), (40.4)

$$s_m = \mathrm{R_I}\gamma^{m-1}\,(\gamma - 1) \qquad \rho_m = \mathrm{R_I}\gamma^{m-\frac{1}{2}}. \qquad (41.4)$$

The number of gravity stations on radius ρ_m is

$$\mathrm{K}_m = 2\pi\rho_m/s_m = 2\pi\gamma^{\frac{1}{2}}/(\gamma - 1), \qquad (41.5)$$

which is the same for all values of m, that is for all rings. The total number of gravity stations is accordingly

$$n = \Sigma \mathrm{K}_m = 2\mathrm{N}\pi\gamma^{\frac{1}{2}}/(\gamma - 1). \qquad (41.6)$$

42—*Case II*—When ε_0 is small compared with $0\cdot 47\ s_m^{\frac{1}{2}}$, we must arrange that

$$\frac{s_m^{\frac{5}{2}}}{\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}} = \text{constant}. \qquad (42.1)$$

This is approximately so if

$$\rho_m = \mathrm{C}\,(\mu^5 + 5\mu^3 + 2\mu) \qquad \text{where} \qquad \mu = m + c \qquad (42.2)$$

$$s_m = 5\mathrm{C}\,(\mu^4 + \tfrac{7}{2}\mu^2 + \tfrac{53}{80}), \qquad (42.3)$$

in which C, c are constants at choice. The numerical coefficients in ρ_m have been so chosen as to make the coefficients of μ^{10}, μ^8, μ^6 in $s_m^{\frac{5}{2}}$ and $\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}$ agree identically, m in (42.2) being replaced by $m + \frac{1}{2}$ and $m - \frac{1}{2}$ in keeping with (40.4). The approximation is good when μ is greater than 2, which in practice will be found necessary. The number of stations in ring m is

$$\mathrm{K}_m = 2\pi\rho_m/s_m = \frac{2\pi}{5}\,\mu\,(1 + 5\mu^{-2} + 2\mu^{-4})\,(1 + \tfrac{7}{2}\mu^{-2} + \tfrac{53}{80}\mu^{-4})^{-1}$$
$$= 1\cdot 26\mu\,(1 + \tfrac{3}{2}\mu^{-2}) = 1\cdot 26\mu + 1\cdot 89\mu^{-1}, \qquad (42.4)$$

so that the total number of stations in N rings is

$$n = 0\cdot 63\left[\mu\,(\mu + 1) + 3\left(0\cdot 577 + \log_e \mu + \frac{1}{2\mu}\ldots\right)\right]_c^{\mathrm{N}+c}$$
$$= 0\cdot 63\,[\mathrm{N}\,(\mathrm{N} + 2c + 1) + 2\cdot 3\log_{10}\{(\mathrm{N} + c)/c\}]. \qquad (42.5)$$

Thus with $\mathrm{N} = 10$, $c = 5$, we find $n = 0\cdot 63\,(210 + 2\cdot 3\log 3) = 133$ which is a reasonably practicable number of stations.

With the help of (42.2) values of ρ_m have been calculated, with related quantities, as set forth in Table V.

TABLE V

$\mu = m + c$	$\frac{\rho_m}{10^4\,C}$	$\frac{\rho_{m+\frac{1}{2}}}{10^4\,C}$	$\frac{s_m}{10^4\,C}$	$K_m = \frac{2\pi\rho_m}{s_m}$ and nearest integer		$[n]^{\mu}_{5}$	Values of 10^4 C making $R_0 \equiv 20°$	Probable error due to one element $= \frac{2}{3}\,C^{\frac{1}{2}}$ for $R_0 \equiv 20°$
5	0·3760	0·5876	—	—		—		
6	0·8868	1·2987	0·7111	7·8	8	8		
7	1·8536	2·5853	1·2966	9·0	9	17		
8	3·5344	4·7456	2·1603	10·3	10	27	—	—
9	6·2712	8·1682	3·4236	11·5	12	39	—	—
10	10·502	13·344	5·176	12·7	13	52	—	—
11	16·773	20·876	7·532	14·0	14	66	—	—
12	25·750	31·496	10·620	15·2	15	81	—	—
13	38·230	46·072	14·576	16·5	16	97	29·4	0·0364
14	55·157	65·624	19·552	17·7	18	115	20·7	0·0303
15	77·628	91·331	25·707	19·0	19	134	14·9	0·0257
16	106·91	124·55	33·22	20·2	20	154	10·9	0·0220
17	144·45	166·81	42·26	21·5	22	176	8·14	0·0189
18	191·88	219·87	53·06	22·7	23	199	6·17	0·0165
19	251·04	285·66	65·79	24·0	24	223	4·75	0·0145
20	324·00	366·36	80·70	25·2	25	248	3·70	0·0128

Using any value of μ in Table V we find

$$s_m^{\frac{5}{2}}/(\rho_{m-\frac{1}{2}}\,\rho_{m+\frac{1}{2}}) = 55{\cdot}9C^{\frac{1}{2}}. \qquad (42.6)$$

We adopt as outer radius of " local " survey region

$$R_0 = \rho_{m+\frac{1}{2}} = a \sin \psi, \qquad \text{with} \qquad \psi = 20°$$
$$= 1354 \text{ miles}. \qquad (42.7)$$

Values of C . 10^4 corresponding to values of μ from 13 to 20 are given in the last column but one of Table V.

From (40.11), ignoring ε_0

$$\text{probable error due to one element} = 0''{\cdot}0254 \times 0{\cdot}47 \times 55{\cdot}9C^{\frac{1}{2}} = 2C^{\frac{1}{2}}/3, \qquad (42.8)$$

values of which are given in the last column of Table V.

Since it has been arranged that each element is equally effective, the p.e. of n elements is clearly

$$2\,(Cn)^{\frac{1}{2}}/3. \qquad (42.9)$$

We have not yet considered what value to take for the inner radius R_I. If $\varepsilon_0 = 1$ mg we can make R_I less than 10 miles; but if $\varepsilon_0 = 2$ mg or more, the p.e. of (42.8) would be an under estimate for the innermost regions. Not many elements are affected and for the moment we shall take R_I approximately $= 10$ miles. Using the values of C from the table, since $R_I = 10^4 \, C \, \rho_{m-\frac{1}{2}}$, we abstract lower values of $\mu = \mu'$ as indicated in Table VI.

TABLE VI

$\mu = N + c$	19		18		17		16		15	
C . 10^4	4·75		6·17		8·14		10·9		14·9	
$\mu' = 0 + c$	6	7	6	7	5	6	5	6	5	6
R_I	6·15	12·3	8·0	15·9	4·8	10·5	6·4	14·1	8·75	19·3
*$[n]_c^{N+c}$	215	206	191	182	176	168	154	146	134	126
p.e. of one element .	0″·0145		0″·0165		0″·0189		0″·0220		0″·0257	
p.e. of region from R_I to R_0	0″·212	0″·208	0″·228	0″·222	0″·251	0″·245	0″·273	0″·266	0″·297	0″·288

* The number includes 50 stations of the world survey, *vide* Table IV, § 38.

43—In the last line of Table VI are given the probable errors for the region between R_I and $R_0 \equiv 20°$. The areas interior and exterior must also be considered. As regards the area within R_I, it is clear that if ε_0 is made very small R_I may be reduced. In some modern gravity work (*e.g.*, the Cambridge expedition to Africa, 1933–4) pendulums are swung simultaneously at headquarters and at the field station, the same time-signals (any Morse message) being recorded in both cases with the pendulum beats. In this way a differential value of gravity is obtained, and the only time errors are those due to the registration and variations of the differential times of signal transmission. It is expected that the differential probable error of observation will not exceed 1 mg. In such a case R_I could safely be taken as small as 5 miles.

For the small region within R_I, we may put

$$g_s = A + B_1 x + B_2 y + \frac{1}{\underline{|2}} (C_1 x^2 + 2C_2 xy + C_3 y^2) \ldots, \quad . \quad . \quad (43.1)$$

in which the numerical coefficients may be determined from the observed values of g. Then

$$\frac{\partial \Delta g_s}{\partial x} = B_1 + C_1 x + C_2 y + \frac{1}{\underline{|2}} (D_1 x^2 + 2D_2 xy + D_3 y^2) \ldots, \quad . \quad (43.2)$$

which may be used directly in (39.6).

It is to be remarked that a steady gradient of one milligal per mile in Δg_s (which is rather unusually large), extending up to a radius of 5 miles, will give rise to a deviation of only 1″ so that the error to be expected in computing the deviation ξ due to this small region, dealt with in the manner laid down, is certain to be very small. It cannot be discussed further with advantage without a specific lay-out of the gravity stations and the observational results, for manifestly much will depend on the degree of smoothness in these latter.

44—We pass now to the region external to R_0 or $\psi = 20°$. From (35.2) we should deduce as probable error due to ring m the value

$$(\text{cosec } 1'' \,.\, P_r \pi H_m)/(2\sqrt{2}\text{GNK}_m^{\frac{1}{2}}),$$

where P_r is the probable error of representation for a *square* of side $s_m = 345$ miles, from (33.11). From (40.6) combined with (39.6) we obtain the value

$$\frac{\text{cosec } 1''}{4\pi G}\left[\frac{\phi}{\rho}\right]_{m-\frac{1}{2}}^{m+\frac{1}{2}} 2\pi\rho_m \,.\, \frac{P'_r}{\sqrt{2}},$$

where P'_r is the probable error of representation for a *side* of length $s_m = \frac{1}{2}P_r$. The two expressions agree approximately

$$\left(\text{since } H_m = \frac{df}{d\psi}\sin\psi, \textit{ vide } (34.4), \text{ and } \left[\frac{\phi}{\rho}\right] = \frac{1}{a}[f] = \frac{\pi}{Na}\frac{df}{d\psi}\right),$$

except that P'_r replaces P_r. This is the result of the cancellation of error in the line integral $\int u dy$ taken round an element of area, *vide* § 40. This could not be taken into account by the first method. Accordingly the value given in (35.2), (35.4) requires to be multiplied by $\frac{1}{2}$. Hence

Probable error of ξ due to region beyond $\psi = 20°$

$$= 0''\cdot 057 \left(\sum_{180°}^{20°} \frac{H_k^2}{K}\right)^{\frac{1}{2}} \quad \ldots \quad (44.1)$$

$$= 0''\cdot 057 \times (10\cdot 90)^{\frac{1}{2}}, \text{ from Table IV}$$

$$= 0''\cdot 188. \quad \ldots \quad (44.2)$$

This is to be combined with one of the errors given in Table VI. For example, with 134 stations between $R_I = 8\cdot 75$ miles and $R_0 \equiv 20°$, the p.e. due to this region is $0''\cdot 297$; while with 191 stations between $R_I = 8\cdot 0$ miles and $R_0 \equiv 20°$, the p.e. is $0''\cdot 288$. Combining these with (44.2) we find

$$\text{Probable error of } \xi \text{ for whole world} = \begin{cases} 0''\cdot 352 \text{ with 134 local stations} \\ 0''\cdot 295 \text{ with 191 local stations} \end{cases}. \quad (44.3)$$

We may estimate the errors which are probable if the world survey extended only up to ψ_w, where ψ_w has various values. Accepting the probable anomaly given by (30.7) as 113 mg in unsurveyed regions = probable error of $(5°)^2 \times 6\cdot 5$, and we must accordingly multiply corresponding values of $\sum_{180}^{\psi_w} \frac{H_k^2}{K}$ from Table IV by $(6\cdot 5)^2$.

In this way we find for a local survey of about 134 stations up to $\psi = 20°$, and world survey thence to ψ_w only

$$\text{Probable error of } \xi \begin{Bmatrix} = 0''\cdot 67 \text{ with } \psi_w = 60° \\ = 0''\cdot 96 \text{ with } \psi_w = 30° \\ = 1''\cdot 26 \text{ with } \psi_w = 20° \end{Bmatrix}. \tag{44.4}$$

A minor point requires mention. If we derive the p.e. of the outer ring for case $\mu = 17$ in Table V (in which $s_m = 42\cdot 26 \times 8\cdot 14 = 344$ miles $\doteqdot 345$ of (33.11) the result is $0''\cdot 0189 \times (22)^{\frac{1}{2}} = 0''\cdot 089$; and if we derive the same quantity from (44.1) applied to ring between 15° and 20° we obtain $0\cdot 057 \times (5\cdot 01)^{\frac{1}{2}} = 0''\cdot 127$. The discrepancy between the two is due to a mean value of ϕ having been used in the former procedure.

Discussion of Probable Errors

45—With the additional local stations it is clear that the p.e. of Δr will be considerably smaller than that estimated in § 38 for a simple world survey, viz., 34 feet. In fact we must form the values of F_k^2/K' (with K' replacing K of Table IV), where K' is the enlarged number of stations. The result is a probable error of 23 feet in Δr.

46—All the estimates of probable error are essentially based on the error of representation, expressed by (29.5). This is in turn based on observational results from a belt of 8° of latitude extending over 14° of longitude; and it is subject to some increase or decrease when data elsewhere are sufficient to admit of similar treatment. It does not appear probable to the writer that any very serious change is likely to be necessary, and if this view is correct, the derived probable errors may be accepted.

The probable error of Δr of 23 feet (§ 45), or approximately $10^{-6}\, a$, is smaller than the probable error of a itself. No doubt the p.e. of a will be subject to considerable reduction when triangulation observations are systematically reduced to the geoid by methods which this paper contemplates. But it is hardly to be expected that the p.e. of a will be reduced below 1 in 10^6, which is indeed of the order of base-line accuracy. An improved p.e. would naturally result from the combination of many base-lines, were these free from systematic error—an assumption which at present can hardly be made. None the less it is of interest to be able to compute the value of Δr, the variation of the geoid from the spheroid of reference, to an equally high or even higher precision. For this purpose a whole world survey involving some 1700 stations is needed; for a consideration of the values of F_k^2/K in Table IV, § 38, shows that the survey should extend sensibly over the whole world.

The probable error of about $0''\cdot 33$ in deviation of the vertical—*vide* (44.3)—is of a suitable order of magnitude to be practically useful. To obtain such a result a serious local survey of about 100 stations within 15° of the origin is needed in addition to the world survey stations, numbering about 1700. It would not be obligatory to extend these latter stations over the whole world if deviation of the vertical only were required—*vide* (44.4) ; but as the latter are required for determination of Δr, the results would naturally be employed for the deviation. We accordingly should have a possible error (equal to three times the probable error) of $1''$—*vide* (44.3). Such precision would be of real value at the origin of any large survey (*cf.* § 27).

When we consider the application to lunar parallaxes and take as example the terrestrial base from Greenwich to Cape Town, which comprises 85° of latitude, we see that liability to error of $1''$ at each end will lead to liability to differential error of $1''\cdot 4$. On the chord $= 2a \sin (85°/2)$ the corresponding proportionate error is $3\cdot 76 \times 10^{-6}$ (= radian measure of $0''\cdot 7 \times \cot 42\frac{1}{2}°$), on account of deviation error. This would have to be combined with error due to errors of a and Δr, the latter being almost negligible in comparison with that due to deviation. There is no doubt that the combined result will be much less than the possible error of any value hitherto employed for such a case.

Reduction of Results of Observation and Practical Computation of Geoidal Anomalies

47—*Use of the Compensated geoid.* To avoid difficulties of matter external to the geoid, some form of compensation must be employed. This is merely a mathematical artifice and does not involve any faith in the actuality of the adopted form of compensation. The combined effects of any matter external to the geoid and of an equivalent amount of matter inside the geoid—distributed as we like—is calculated ; both on the value of gravity at any point and on the height of the geoid. The result is a set of compensated values of gravity which is responsible for the form of the compensated geoid.

The separation of geoid and compensated geoid is easily calculated when the topography is known.* The reduction of gravity observations, including compensation according to the Hayford hypothesis, is well understood. The fact that many gravity results have already been so reduced makes it convenient to adopt this form of compensation. For the present purpose, in addition a height correction for the amount of separation of the geoid (from which heights of pendulum stations are ordinarily given) and compensated geoid is required ; for our data are to be applied to the compensated geoid. The height correction in all cases may be derived best by means of equation (17.4).

We have tacitly assumed hitherto that the compensation correction will remove all effects of matter external to the real geoid ; but we now require that the same be true of the compensated geoid. Whether this is true or not depends on the nature of the compensation. If for each external particle, at height h above the geoid, we assumed an equal negative compensatory particle at depth h below the geoid, then

* de Graaff Hunter and Bomford, ' Bull. géod. int.' No. 29 (1931).

the compensated geoid would be coincident with the real geoid, and our intention would be properly fulfilled. But for practical reasons we propose the use of compensation after the hypothesis of HAYFORD ; and here the compensation of a particle at height h is distributed vertically through a depth much greater than h. The removal of the effect of the particle is to lower the geoid by an amount greater than the compensation will raise it. Accordingly the compensated geoid will in land areas ordinarily lie somewhat lower than the real geoid. When this occurs a further, but small, correction is required to remove the effects of this matter external to the compensated geoid. In India, the separation of compensated geoid and real geoid is generally less than 10 feet* and the correction is negligible. In ocean areas, the need for this additional correction does not arise, for the compensated geoid will in general lie without the geoid. It is to be remembered that the final act will be to restore the several corrections for compensation, so that their detailed nature is not of ultimate importance.

48—*Practical computation of geoidal anomalies*—Having reduced all data as just indicated, computation of compensated geoidal rise is given by (34.5) so far as the general world survey results are concerned ; while for the more detailed local survey, values of F_k of (34.4) appropriate to the regions indicated in Table II, § 42, will have to be computed.

For the values of tilt, the formula—*vide* (39.6), (40.10), (40.8)—is

$$\begin{matrix}\delta\eta'' \\ \delta\xi''\end{matrix} = +\,0''\cdot0167\,\Sigma\left[\frac{\phi}{\rho}\right]_{m+\frac{1}{2}}^{m-\frac{1}{2}} \rho_m\,.\,2\pi\,.\,\Delta h_m,$$

or

$$\begin{matrix}\delta\eta'' \\ \delta\xi''\end{matrix} = +\,0''\cdot105\,\Sigma\,(f_{m-\frac{1}{2}} - f_{m+\frac{1}{2}})\,\frac{\rho_m}{a}\,\Delta h_m, \quad . \quad . \quad . \quad . \quad . \quad (48.1)$$

where Δh_m = average value of $\Delta g_s \begin{matrix}\cos \\ \sin\end{matrix} \chi$ along circle ρ_m, *vide* (34.2). Values of ρ_m are given in Table V, § 42 ; those of f in Table IV, § 38. For the region beyond $\psi = 20°$, equation (34.5) may be used within proper limits. The combined result for region outside a radius of about 8 miles is

$$\begin{matrix}\eta'' \\ \xi''\end{matrix} = 0''\cdot105\,\overset{20°}{\Sigma}\,(f_{m-\frac{1}{2}} - f_{m+\frac{1}{2}})\,\frac{\rho_m}{a}\,\Delta h_m + 0''\cdot0092\sum_{k=5}^{k=36}\Delta h_k\,.\,H_k, \qquad (48.2)$$

taking N = 36. For the portion within the small radius of 8 miles, *see* § 43.

49—The final operation will be to restore the compensation corrections to the values of elevation and tilt ($\Delta\rho$, η'', ξ''), that is the separation of the compensated geoid and real geoid, and the corrections to deviations of the vertical. Our results will then display the actual geoid. The only residual assumption will then be in our

* *Vide* " Geodetic Report," ' Survey of India,' vol. 5, Chart IX.

process relating to the internal density of the portion of the earth external to the geoid ; a matter which may reasonably be considered separately from those dealt with above.

Conclusions

50—It has been shown that the form of the sea-level surface of the earth—the real geoid—can easily be derived from the form of a compensated geoid. The law of external potential of a rotating, gravitational solid, whose boundary is an equipotential and differs slightly from a spheroid, has been deduced ; and from this the departures of a compensated geoid from a reference spheroid, suitably chosen, has been expressed in terms of the departures of compensated gravity from the values given by a standard formula representing the gravity of this reference spheroid, now considered a solid rotating body. The deduction has been made without any assumption as to the *form* of the potential.

The distribution of gravity stations and their number, designed to give an adequate precision in the form of the compensated geoid which it is proposed to deduce therefrom, has been considered. There are three main objects for this determination :

(*a*) To determine the elements at the origin of a geodetic survey, that is, the relative location as regards height and tilt of the compensated geoid (and thence of the real geoid) and of the adopted reference spheroid. By this means only is it possible to refer all such unconnected surveys to a unique reference system.

(*b*) To determine the same elements at two widely separated points on the earth with a view to precise computation of a long terrestrial base, destined to fix the scale of the universe by means of lunar parallax observations.

(*c*) To determine the general figure of the earth for general purposes.

For these objects it is necessary to have a general distribution of gravity observations over the entire earth, which will be sufficient to meet (*c*) and go part way to meet (*a*), (*b*). In the vicinity of those special points—origins of survey and terminals of great terrestrial bases—contemplated for (*a*), (*b*), additional local distribution of gravity stations is required in each case. It has been found that for the general distribution, one station for each region of the same area as a quasi-square of 5° of latitude by 5° of longitude (at the equator) will be of great practical utility. This implies in all 1654 stations, some of which have already been formed. For each local distribution, additional stations, to the number of about 100, are required, properly distributed within 15° of the origin concerned. The probable error of representation, by which is implied the probable difference between the value of gravity at a point and the mean value for the vicinity, has been investigated and a semi-empirical expression has been derived from which its value may be assessed for regions of various magnitudes. With the help of this it has been possible to assess the probable errors which will arise in computing geoidal form. It has also been possible to consider the *most economical* local distribution of stations. If Δr is the separation of compensated geoid and spheroid of reference, and η'', ξ'' are components of the deviation of the geoidal vertical from the spheroidal vertical, then

§ 38 p.e. of Δr, derived from the general distribution of 1654 stations $= \pm 34$ feet $= \pm 10\cdot 4$ metres.

§ 45 p.e. of Δr, derived from the 1654 general and 84 local stations $= \pm 23$ feet $= \pm 7\cdot 0$ metres.

(44.3) p.e. of η'' or ξ'', derived from 84 local stations and 1654 general stations $= \pm 0''\cdot 35$.

(44.4) Do. with the general distribution carried only to 60° from origin $= \pm 0''\cdot 67$.

(44.4) Do. with the general distribution carried only to 20° from origin $= \pm 1''\cdot 26$.

These results show that the proposed number and distribution of gravity stations yield results of precision adequate for the purpose for which they are intended. It is clear that for geoidal elevation the general distribution of stations must be practically complete over the whole earth but that no special local distribution is necessary; while for geoidal tilt the local distribution is of primary importance and the general distribution might conceivably be limited to within 60° of the origin; the probable error would in that case be doubled.

In the application to terrestrial bases for use with parallax observations, it has been found (§ 46) that the probable scale error due to error in geoidal elevation and tilt is $1\cdot 25 \times 10^{-6}$—a quantity distinctly less than the associated error due to imperfect determination of the mean radius, a. The latter determination, however, is susceptible of considerable improvement when the results of all geodetic surveys are reduced with due regard to geoidal form, as becomes possible when the methods now advocated are put into practice.